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ON CONGESTION INTERDEPENDENCE AND URBAN TRANSIT FARES. SOME EXTENSIONS

par

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Mémoire présenté en vue de

l'obtention du grade de

Maître en Sciences Economiques

Directeur: Maurice Marchand

Janvier 1982

ON CONGESTION INTERDEPENDENCE AND URBAN TRANSIT FARES. SOME EXTENSIONS

by

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INTRODUCTION

1. It is classic that use of transportation modes in congested urban streets creates costs for other users of them which are not taken into account in individual transport decisions. These congestion costs refer to the increased consumption of marketable and non-marketable inputs[++] necessary to perform travels between two points belonging to the urban area at peak periods. It is also classic that the first-best Pigovian solution to this externality problem lies in the use of congestions taxes which increase the price of transportation services until the tax inclusive price faced by users is equated to the marginal cost of providing these services inclusive of the congestion costs imposed on others.

2. In practice, however, feasibility constraints due e.g. to technical and political difficulties in introducing optimal taxation prevent reaching first-best Pareto optimality and second-best maxima are all that can be obtained. In this line are the works we now briefly review and also the present paper. Lévy-Lambert(1968) studies the case of two perfectly substitutable facilities (in our case two modes of transport -public bus and private automobile services- both sharing the same scarce street space) and concludes that bus fares should fall short of the marginal social cost of the service if no toll is charged to private vehicles for street use at the period of congestion. Marchand (1968) derives Lévy-Lambert's result from a general equilibrium framework and examines its implications for income redistribution. Sherman(1971), following the

[+] I am indebted to Professor Maurice Marchand for suggesting this paper and for many helpful suggestions. All remaining errors are of course mine.

[++] Among marketable inputs we can find gasoline, oil, maintenance, wear and tear, etc. while non-marketable inputs include time spent and injuries suffered by transportation modes' users. Pollution is a byproduct of congestion bearing upon all urban dwelers.

methodology developed by Marchand, extends the analysis to allow interdependence in congestion at peak time by the two alternative transportation modes. He first studies the conditions under which a second-best direct tax impinging upon peak bus fares implies a subsidy to the public bus company, provided automobiles are not subject to direct taxes. Then he investigates the case where, besides the direct tax on peak bus fares, automobiles can be indirectly taxed through an input (gasoline) tax prevalent at both peak and off-peak periods. In both cases the assumptions are made that off-peak time is free of congestion, that travel demands between periods are independent, and that public authorities can implement lump sum transfers of income to consumers. Sherman also gives some reflections on the bus company's budgetary balance. Bertrand (1977), disregarding the question of income redistribution and the explicit examination of peak and off-peak transit pricing, extends the discussion of second-best direct optimal taxation to a multimode traffic network.

3. The present paper provides some extensions to Sherman's work. All models we analyze explicitly assume a two (peak, off-peak) periods demand structure for each of the two transportation modes we consider (public bus and private automobile). Congestion "intraperiod" interdependence is also postulated for all of them. In Model I we characterize the optimal differentiated (peak, off-peak) bus fare structure when public authorities have at their disposal the following instruments: two bus fares, a tax on gasoline bearing upon private motorists independent of the time period, and lump sum transfers of income for consumers. In this model we also limit the bus company to a budget constraint. In Model II we delete the budget constraint. In Model III we study the case in which the bus firm is subject to fix a uniform fare between periods but is free of budget constraint. In Model IV, we add again a budget constraint to the bus company. Finally, in Model V we introduce equity considerations into the discussion by postulating a uniform lump-sum transfer of income to consumers, to the study of a differentiated bus fare structure with a budget constraint.

MODEL I. A DIFFERENTIATED BUS FARE STRUCTURE WITH A BUDGET CONSTRAINT.

We consider an economy where each of n consumers (i=1,...,n) derives satisfaction from travelling by two different modes at two different periods of time, private automobile passnger miles at peak period (t^i_{a1}), private automobile passenger miles at off-peak period (t^i_{a2}), public bus passenger miles at peak period (t^i_{b1}) and public bus passenger miles at off-peak period (t^i_{b2}), and also from consuming units of a composite commodity (x^i) - the numeraire.

Let us make the following assumptions:

A1. Utility functions are quasiconcave, continuous, and twice differentiable:

(1) $U^i = U^i(t^i_{a1}, t^i_{a2}, t^i_{b1}, t^i_{b2}, x^i),\ U^i \in C^2$ (i=1,...,n)

A2. Average input requirements of gasoline -the only transit service input we consider in this part- to produce units t_{ak} and t_{bk} (k=1,2) are represented by variables g_{ak} and g_{bk}, and congestion "intraperiod" interdependence is postulated:

(2) $g_{ak}=g_{ak}(t_{ak},t_{bk}),\ g_{ak} \in C^2,\ \partial g_{ak}/\partial t_{ak} \geqslant 0,\ \partial g_{ak}/\partial t_{bk} \geqslant 0,$ k=1,2;

(3) $g_{bk}=g_{bk}(t_{ak},t_{bk}),\ g_{bk} \in C^2,\ \partial g_{bk}/\partial t_{ak} \geqslant 0,\ \partial g_{bk}/\partial t_{bk} \geqslant 0,$ k=1,2;

where $t_{ak} = \Sigma_i t^i_{ak}$, $t_{bk} = \Sigma_i t^i_{bk}$ are, respectively, the global demand for private transit services at period k, and the global demand for public (bus) transit services at period k, k= 1,2.

A3. Production possibilities of the economy are constrained by the concave, twice continuously differentiable transformation function:

(4) $f(g,x) = 0,\ f \in C^2$

where g stands for gasoline and x for composite commodity.

A4. Competition is postulated on other markets, so that we obtain equality between the marginal rate of transformation and the (producer) price ratio for g and x:

(5) $-(dx/dg) = P_g/P_x = P_g$

since we normalize by setting $P_x = 1$.

A5. The bus company -which plays the role of a social planner- has at its disposal the following instruments in order to reach an efficient allocation of resources:

1/ a tax θ on gasoline purchased by private motorists, independent of the time period,

2/ two bus fares: P_{b1} (peak) and P_{b2} (off-peak), and

3/ lump sum transfers of income, y^i to consumers.

A6. The consumer private price system (P_{a1}, P_{a2}, P_{b1} and P_{b2} per unit, respectively, of t_{a1}, t_{a2}, t_{b1} and t_{b2}) is defined as follows:

(6) $P_{a1} = (P_g + \theta) g_{a1}$,

(7) $P_{a2} = (P_g + \theta) g_{a2}$,

(8) $P_{b1} = P_g g_{b1} + p_{b1}$, and

(9) $P_{b2} = P_g g_{b2} + p_{b2}$;

where p_{b1} and p_{b2} stand for optimal "tolls" to be charged on public bus fares P_{b1} and P_{b2}, respectively[1].

We finally assume

A7. The public bus company is subject to a budget constraint, B. This can be formalized as follows[2]:

(10) $t_{b1} p_{b1} + t_{b2} p_{b2} = B$,

where B stands for a fixed profit requirement.

With a consumer price vector $P = (P_{a1}, P_{a2}, P_{b1}, P_{b2})$, x as numeraire, and y^i as the consumer i's income, budget constraints are:

(11) $P_{a1} t_{a1}^i + P_{a2} t_{a2}^i + P_{b1} t_{b1}^i + P_{b2} t_{b2}^i + x^i = y^i$ (i=1,...,n)

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1. It is instructive to look at private prices as total private costs per unit of transport service.

2. Defining bus firm's total revenue as $TR_b \equiv \Sigma_{k=1}^2 t_{bk} P_{bk}$, and its total variable costs as $TVC_b \equiv P_g \Sigma_{k=1}^2 t_{bk} g_{bk}$, its total gross profit becomes

$$\Pi_b \equiv TR_b - TVC_b = \Sigma_{k=1}^2 t_{bk} p_{bk}.$$

When a budget constraint, B, is imposed upon it, $\Pi_b = B$, and (10) follows.

Maximization of (1) subject to (11) for each i requires

$$\partial U^i/\partial t^i_{a1} = \lambda^i p_{a1} \qquad (i=1,\ldots,n) \tag{12}$$

$$\partial U^i/\partial t^i_{a2} = \lambda^i p_{a2} \qquad (i=1,\ldots,n) \tag{13}$$

$$\partial U^i/\partial t^i_{b1} = \lambda^i p_{b1} \qquad (i=1,\ldots,n) \tag{14}$$

$$\partial U^i/\partial t^i_{b2} = \lambda^i p_{b2} \qquad (i=1,\ldots,n) \tag{15}$$

$$\partial U^i/\partial x^i = \lambda^i \qquad (i=1,\ldots,n) \tag{16}$$

where λ^i is the marginal utility of income for consumer i. These conditions, together with (11) yield demand functions

$$t^i_{a1} = t^i_{a1}(P,y^i) \qquad (i=i,\ldots,n) \tag{17}$$

$$t^i_{a2} = t^i_{a2}(P,y^i) \qquad (i=1,\ldots,n) \tag{18}$$

$$t^i_{b1} = t^i_{b1}(P,y^i) \qquad (i=1,\ldots,n) \tag{19}$$

$$t^i_{b2} = t^i_{b2}(P,y^i) \qquad (i=1,\ldots,n) \tag{20}$$

$$x^i = x^i(P,y^i) \qquad (i=1,\ldots,n) \tag{21}$$

The analysis we undertake is of second-best Pareto optimality nature because of the following reason: gasoline need not be purchased right at the time it is used so that θ cannot be varied by time of day. Planner's problem is then to select the triplet (θ,p_{b1},p_{b2}) in order to minimize misallocation[3].

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3. Recall from the introduction that misallocation in this economy arises because consumers of transit services do not take into account congestion costs they impose on others. These congestion costs take the form, in our model, of an increased consumption of gasoline per unit of transit service (see also assumption A2). First best optimality could be achieved if our planner could directly modify consumers' behaviour by differentiating θ according to peak and off-peak periods. Feasibility condition stated above prevents him from doing so.

The conditions for a second-best Pareto optimum are obtained by maximizing a welfare function defined by a linear combination (with arbitrary weights) of the individual utility functions, subject to appropriate constraints, namely

$$\text{(22)} \quad \underset{\substack{(P_{b1},P_{b2},P_{b1},P_{b2},P_g,\\ \theta,y^i,t_{a1},t_{a2},t_{b1},t_{b2})}}{\text{Max}} \; W \equiv \Sigma_{i=1}^{n}\beta^i U^i[t_{a1}^i(P,y^i),t_{a2}^i(P,y^i),t_{b1}^i(P,y^i),t_{b2}^i(P,y^i),x^i(P,y^i)]$$

subject to

$$\text{(23)} \quad P_{b1} = P_g g_{b1}(t_{a1},t_{b1}) + p_{b1} \qquad (\mu_{b1})$$

$$\text{(24)} \quad P_{b2} = P_g g_{b2}(t_{a2},t_{b2}) + p_{b2} \qquad (\mu_{b2})$$

$$\text{(25)} \quad \Sigma_i t_{a1}^i = t_{a1} \qquad (\tau_{a1})$$

$$\text{(26)} \quad \Sigma_i t_{a2}^i = t_{a2} \qquad (\tau_{a2})$$

$$\text{(27)} \quad \Sigma_i t_{b1}^i = t_{b1} \qquad (\tau_{b1})$$

$$\text{(28)} \quad \Sigma_i t_{b2}^i = t_{b2} \qquad (\tau_{b2})$$

$$\text{(29)} \quad t_{a1}g_{a1}+t_{a2}g_{a2}+t_{b1}g_{b1}+t_{b2}g_{b2} = g(P_g) \qquad (\gamma)$$

$$\text{(30)} \quad \Sigma_i x^i = x(P_g) \qquad (\chi)$$

$$\text{(31)} \quad t_{b1}p_{b1} + t_{b2}p_{b2} = B \qquad (\rho)$$

where Greek letters at the right are Lagrange multipliers associated with the constraints. These constraints also imply

$$\Sigma_i y^i = x + P_g g + \theta(g_{a1}t_{a1}+g_{a2}t_{a2}) + p_{b1}t_{b1} + p_{b2}t_{b2}$$

indicating that the proceeds from gasoline tax on private users, and the sum of "tolls" collected are redistributed to consumers. The first order conditions for a maximum (in addition to constraints (23) through (31)) are:

$$\text{(32)} \quad -\mu_{b1} + \rho t_{b1} = 0$$

$$\text{(33)} \quad -\mu_{b2} + \rho t_{b2} = 0$$

(34) $$-\Sigma_i\beta^i\lambda^i t^i_{b1} + \mu_{b1} - \tau_{a1}\Sigma_i(\partial t^i_{a1}/\partial P_{b1}) - \tau_{a2}\Sigma_i(\partial t^i_{a2}/\partial P_{b1}) - \tau_{b1}\Sigma_i(\partial t^i_{b1}/\partial P_{b1})$$
$$- \tau_{b2}\Sigma_i(\partial t^i_{b2}/\partial P_{b1}) - \chi\Sigma_i(\partial x^i/\partial P_{b1}) = 0$$

(35) $$-\Sigma_i\beta^i\lambda^i t^i_{b2} + \mu_{b2} - \tau_{a1}\Sigma_i(\partial t^i_{a1}/\partial P_{b2}) - \tau_{a2}\Sigma_i(\partial t^i_{a2}/\partial P_{b2}) - \tau_{b1}\Sigma_i(\partial t^i_{b1}/\partial P_{b2})$$
$$- \tau_{b2}\Sigma_i(\partial t^i_{b2}/\partial P_{b2}) - \chi\Sigma_i(\partial x^i/\partial P_{b2}) = 0$$

(36) $$-\Sigma_i\beta^i\lambda^i[t^i_{a1}g_{a1} + t^i_{a2}g_{a2}] - \mu_{b1}g_{b1} - \mu_{b2}g_{b2} - \tau_{a1}\Sigma_i[(\partial t^i_{a1}/\partial P_{a1})g_{a1} +$$
$$+ (\partial t^i_{a1}/\partial P_{a2})g_{a2}] - \tau_{a2}\Sigma_i[(\partial t^i_{a2}/\partial P_{a1})g_{a1} + (\partial t^i_{a2}/\partial P_{a2})g_{a2}]$$
$$- \tau_{b1}\Sigma_i[(\partial t^i_{b1}/\partial P_{a1})g_{a1} + (\partial t^i_{b1}/\partial P_{a2})g_{a2}] - \tau_{b2}\Sigma_i[(\partial t^i_{b2}/\partial P_{a1})g_{a1} +$$
$$+ (\partial t^i_{b2}/\partial P_{a2})g_{a2}] + \gamma(\partial g/\partial P_g) + \chi(\partial x/\partial P_g)$$
$$- \chi\Sigma_i[(\partial x^i/\partial P_{a1})g_{a1} + (\partial x^i/\partial P_{a2})g_{a2}] = 0$$

(37) $$-\Sigma_i\beta^i\lambda^i[t^i_{a1}g_{a1} + t^i_{a2}g_{a2}] - \tau_{a1}\Sigma_i[(\partial t^i_{a1}/\partial P_{a1})g_{a1} + (\partial t^i_{a1}/\partial P_{a2})g_{a2}]$$
$$- \tau_{a2}\Sigma_i[(\partial t^i_{a2}/\partial P_{a1})g_{a1} + (\partial t^i_{a2}/\partial P_{a2})g_{a2}]$$
$$- \tau_{b1}\Sigma_i[(\partial t^i_{b1}/\partial P_{a1})g_{a1} + (\partial t^i_{b1}/\partial P_{a2})g_{a2}]$$
$$- \tau_{b2}\Sigma_i[(\partial t^i_{b2}/\partial P_{a1})g_{a1} + (\partial t^i_{b2}/\partial P_{a2})g_{a2}]$$
$$- \chi\Sigma_i[(\partial x^i/\partial P_{a1})g_{a1} + (\partial x^i/\partial P_{a2})g_{a2}] = 0$$

(38) $$\beta^i\lambda^i - \tau_{a1}(\partial t^i_{a1}/\partial y^i) - \tau_{a2}(\partial t^i_{a2}/\partial y^i) - \tau_{b1}(\partial t^i_{b1}/\partial y^i)$$
$$- \tau_{b2}(\partial t^i_{b2}/\partial y^i) - \chi(\partial x^i/\partial y^i) = 0 \qquad (i=1,\dots,n)$$

(39) $$-\mu_{b1}P_g(\partial g_{b1}/\partial t_{a1}) + \tau_{a1} - \gamma[g_{a1} + t_{a1}(\partial g_{a1}/\partial t_{a1}) + t_{b1}(\partial g_{b1}/\partial t_{a1})] = 0$$

(40) $$-\mu_{b2}P_g(\partial g_{b2}/\partial t_{a2}) + \tau_{a2} - \gamma[g_{a2} + t_{a2}(\partial g_{a2}/\partial t_{a2}) + t_{b2}(\partial g_{b2}/\partial t_{a2})] = 0$$

$$(41)\quad -\mu_{b1}P_g(\partial g_{b1}/\partial t_{b1}) + \tau_{b1} - \gamma[g_{b1}+t_{b1}(\partial g_{b1}/\partial t_{b1})+t_{a1}(\partial g_{a1}/\partial t_{b1})] + \rho p_{b1} = 0$$

$$(42)\quad -\mu_{b2}P_g(\partial g_{b2}/\partial t_{b2}) + \tau_{b2} - \gamma[g_{b2}+t_{b2}(\partial g_{b2}/\partial t_{b2})+t_{a2}(\partial g_{a2}/\partial t_{b2})] + \rho p_{b2} = 0$$

Remark.- In order to derive conditions (34) through (37), and (38), two properties of indirect utility functions

$$V^i(P,y^i) \equiv \text{Max } U^i[t^i_{a1}(P,y^i),t^i_{a2}(P,y^i),t^i_{b1}(P,y^i),t^i_{b2}(P,y^i),x^i(P,y^i)]$$

have been exploited, namely[4]:

$$\text{i)}\quad \partial V^i/\partial P_k = -\lambda^i t^i_k, \qquad k = a1,a2,b1,b2$$

$$\text{ii)}\quad \partial V^i/\partial y^i = \lambda^i \qquad (i=1,\dots,n)$$

First order conditions derived above can be used to obtain optimal values for our planner's control variables p_{b1}, p_{b2} and θ. To simplify the picture we first substitute (38) into (34) through (37), taking advantage of conditions (32) and (33). Then we define Slutsky compensated effects as

$$(43)\quad \begin{aligned} \partial\hat{t}_j/\partial P_k &\equiv \Sigma_i[(\partial t^i_j/\partial P_k) + (\partial t^i_j/\partial y^i)t^i_k] && j,k=a1,a2,b1,b2 \\ \partial\hat{x}/\partial P_k &\equiv \Sigma_i[(\partial x^i/\partial P_k) + (\partial x^i/\partial y^i)t^i_k] && k=a1,a2,b1,b2 \end{aligned} \qquad (i=1,\dots,n)$$

For example, if we take j=k=a1, $\partial\hat{t}_j/\partial P_k$ represents the effect on demand for auto passenger miles at peak time of a compensated change in its price. Finally we make use of the following fact upon pure substitution effects

$$(44)\quad P_{a1}(\partial\hat{t}_{a1}/\partial P_k) + P_{a2}(\partial\hat{t}_{a2}/\partial P_k) + P_{b1}(\partial\hat{t}_{b1}/\partial P_k) + P_{b2}(\partial\hat{t}_{b2}/\partial P_k)$$

$$(\partial\hat{x}/\partial P_k) = 0, \qquad k=a1,a2,b1,b2$$

The above system of equations denotes the basic property that the sum of Slutsky terms weighted by the prices adds up to zero.

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4. Note <u>en passant</u> that combining i) and ii) yield Roy's identities

$$t^i_k = -[(\partial V^i/\partial P_k)/(\partial V^i/\partial y^i)], \; k = a1,a2,b1,b2; \quad i=1,\dots,n$$

The resulting expressions for (34) through (37) are after some manipulation:

(45) $$[(\tau_{a1}/\chi)-P_{a1}](\partial\hat{t}_{a1}/\partial P_{b1}) + [(\tau_{a2}/\chi)-P_{a2}](\partial\hat{t}_{a2}/\partial P_{b1}) + [(\tau_{b1}/\chi)-P_{b1}](\partial\hat{t}_{b1}/\partial P_{b1}) + [(\tau_{b2}/\chi)-P_{b2}](\partial\hat{t}_{b2}/\partial P_{b1}) = (\rho/\chi)t_{b1}$$

(46) $$[(\tau_{a1}/\chi)-P_{a1}](\partial\hat{t}_{a1}/\partial P_{b2}) + [(\tau_{a2}/\chi)-P_{a2}](\partial\hat{t}_{a2}/\partial P_{b2}) + [(\tau_{b1}/\chi)-P_{b1}](\partial\hat{t}_{b1}/\partial P_{b2}) + [(\tau_{b2}/\chi)-P_{b2}](\partial\hat{t}_{b2}/\partial P_{b2}) = (\rho/\chi)t_{b2}$$

(47) $$[(\tau_{a1}/\chi)-P_{a1}][(\partial\hat{t}_{a1}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{a1}/\partial P_{a2})g_{a2}] + [(\tau_{a2}/\chi)-P_{a2}][(\partial\hat{t}_{a2}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{a2}/\partial P_{a2})g_{a2}] + [(\tau_{b1}/\chi)-P_{b1}][(\partial\hat{t}_{b1}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{b1}/\partial P_{a2})g_{a2}] + [(\tau_{b2}/\chi)-P_{b2}][(\partial\hat{t}_{b2}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{b2}/\partial P_{a2})g_{a2}] = (1/\chi)[-\rho(t_{b1}g_{b1}+t_{b2}g_{b2}) + \gamma(\partial g/\partial P_g) + \chi(\partial x/\partial P_g)]$$

(48) $$[(\tau_{a1}/\chi)-P_{a1}][(\partial\hat{t}_{a1}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{a1}/\partial P_{a2})g_{a2}] + [(\tau_{a2}/\chi)-P_{a2}][(\partial\hat{t}_{a2}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{a2}/\partial P_{a2})g_{a2}] + [(\tau_{b1}/\chi)-P_{b1}][(\partial\hat{t}_{b1}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{b1}/\partial P_{a2})g_{a2}] + [(\tau_{b2}/\chi)-P_{b2}][(\partial\hat{t}_{b2}/\partial P_{a1})g_{a1} + (\partial\hat{t}_{b2}/\partial P_{a2})g_{a2}] = 0$$

It should be noted that the LHS's of equations (47) and (48) coincide thanks to the fact that $\partial P_{ak}/\partial P_g = \partial P_{ak}/\partial\theta = g_{ak}$, k=1,2. This in turn implies the nullity of the RHS of equation (47), or what is equivalent

$$\rho(t_{b1}g_{b1}+t_{b2}g_{b2}) = \gamma(\partial g/\partial P_g) + \chi(\partial x/\partial P_g)$$

since χ is assumed finite and different from zero.

Now, since by assumption A5. the productive sector behaves competitively, we can write[5] $(\partial x/\partial P_g) = - P_g(\partial g/\partial P_g)$ and substitute this fact into the above expression to get

$$(49) \quad \gamma/\chi = (\rho/\chi)(t_{b1}g_{b1}+t_{b2}g_{b2})(\partial g/\partial P_g)^{-1} + P_g$$

We finally derive the last important set of equations by substituting (49) into conditions (39) through (42), making use of (32) and (33), to obtain

$$(50) \quad \tau_{a1}/\chi = (\rho/\chi)P_g t_{b1}(\partial g_{b1}/\partial t_{a1}) + [(\rho/\chi)(t_{b1}g_{b1}+t_{b2}g_{b2})(\partial g/\partial P_g)^{-1} + P_g]$$
$$[g_{a1}+t_{a1}(\partial g_{a1}/\partial t_{a1})+t_{b1}(\partial g_{b1}/\partial t_{a1})]$$

$$(51) \quad \tau_{a2}/\chi = (\rho/\chi)P_g t_{b2}(\partial g_{b2}/\partial t_{a2}) + [(\rho/\chi)(t_{b1}g_{b1}+t_{b2}g_{b2})(\partial g/\partial P_g)^{-1} + P_g]$$
$$[g_{a2}+t_{a2}(\partial g_{a2}/\partial t_{a2})+t_{b2}(\partial g_{b2}/\partial t_{a2})]$$

$$(52) \quad \tau_{b1}/\chi = (\rho/\chi)[P_g t_{b1}(\partial g_{b1}/\partial t_{b1})-p_{b1}] + [(\rho/\chi)(t_{b1}g_{b1}+t_{b2}g_{b2})(\partial g/\partial P_g)^{-1} + P_g]$$
$$[g_{b1}+t_{b1}(\partial g_{b1}/\partial t_{b1})+t_{a1}(\partial g_{a1}/\partial t_{b1})]$$

$$(53) \quad \tau_{b2}/\chi = (\rho/\chi)[P_g t_{b2}(\partial g_{b2}/\partial t_{b2})-p_{b2}] + [(\rho/\chi)(t_{b1}g_{b1}+t_{b2}g_{b2})(\partial g/\partial P_g)^{-1} + P_g]$$
$$[g_{b2}+t_{b2}(\partial g_{b2}/\partial t_{b2})+t_{a2}(\partial g_{a2}/\partial t_{b2})]$$

Having taken advantage of the RHS of equation (47), this equation can be deleted since its LHS is already present in equation (48) -the condition on θ-, and its reproduction would be redundant. The model, therefore, is closed by equations (45),(46),(48) and (50) through (53). Our next task then is to give an interpretation of such a system keeping in mind that our final goal is to find the triplet (θ, p_{b1}, p_{b2}).

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5. This relationship derives from production theory in perfect competition. Suppose a producer considers the problem of maximizing profits subject to the technological restrictions represented by transformation function, i.e., $\max_{(x,g)} x + P_g g \quad \text{s.t. } f(x,g) = 0.$

From first order conditions $P_g = f'_g/f'_x = - dx/dg$ (the last expression coming from setting $df = 0$), and taking advantage of functions $x = x(P_g)$ and $g = g(P_g)$, it is immediate that $(\partial x/\partial P_g) = - P_g(\partial g/\partial P_g)$.

Let us comment first on equations (45), (46) and (48), which show a strong resemblance with those found in optimal taxation literature. We begin with the first of them, which says :

i) The sum of divergences between social and private marginal costs weighted by (compensated) price effects adds up to bus firm's supply of transit services at peak period, t_{b1}, times the shadow price of a marginal unit of budget requirement B. Therefore, the extent of deviations from marginal social cost pricing depends on the budget constraint.

ii) "The LHS is the change in the demand for good (t_{b1}) that would result if consumers were compensated to stay on the same indifference curves and the derivatives of the compensated demand curves were constant... for small taxes ... [$\ell_j \equiv (\tau_j/\chi) - P_j$; $j = a1, a2, b1, b2$] the optimal tax structure involves an equal proportionate movement along the compensated demand curve for all goods" (since ρ/χ is independent of t_{b1}). (Atkinson & Stiglitz, 1980, p.373) (Brackets are mine.)

iii) "A small increase in shadow taxes ($\ell_j \equiv (\tau_j/\chi) - P_j$, $j = a1, a2, b1, b2$), would lead to a change in compensated demands which is proportional for any commodity j to the ... supply of" (t_{b1}). (Guesnerie, 1980, p.64) (Brackets are mine.)

All three points i), ii) and iii) are of application to (46). Equation (48), which we refer to as the condition on θ, can be more easily interpreted if we restate it as

$$\sum_j [(\tau_j/\chi) - P_j](\partial\hat{t}_j/\partial\theta) = 0 \qquad j = a1, a2, b1, b2$$

indicating that the sum of relative divergences between social and private marginal costs weighted by the derivatives of the compensated demand curves with respect to θ adds up to zero. This is so because θ does not depend on ρ/χ.

The economic meaning of equations (50) through (53) may be seen from looking e.g. at the first of them. Let us write it down as

$$\tau_{a1}/\chi = P_g(\Delta g/\Delta t_{a1}) + (\rho/\chi)[P_g t_{b1}(\partial g_{b1}/\partial t_{a1}) + g_b(\partial P_g/\partial g)(\Delta g/\Delta t_{a1})]$$

where:

$$g_b = t_{b1}g_{b1} + t_{b2}g_{b2}$$

stands for total consumption of gasoline by the bus company,

$$(\Delta g/\Delta t_{a1}) \equiv [g_{a1} + t_{a1}(\partial g_{a1}/\partial t_{a1}) + t_{b1}(\partial g_{b1}/\partial t_{a1})] > 0$$

represents the (marginal) rate at which total consumption of gasoline, g, increases per infinitesimal increase in t_{a1}, and it is strictly positive (see also assumption A2);

$$(\rho/\chi) \equiv (\Delta x/\Delta B)\Big|_W > 0$$

can be viewed as the value, in terms of numeraire, of a marginal increase in bus firm's budget requirement, B, and we postulate it positive[6].

Let us define now

$$MC_{a1} = P_g(\Delta g/\Delta t_{a1})$$

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6. We are assuming here that the public firm is constrained to make a profit above the level required for first best optimality - say B^*. In such a case $\rho > 0$ indicating that an increase in B is equivalent to a withdrawal of resources from the system implying an efficiency loss. To check the sign of the equality

$$(\rho/\chi) = (\Delta x/\Delta B)\Big|_W > 0$$

(i.e., in order to keep welfare constant an increase in B must be accompanied by a positive change in x), let us assume that an exogenous amount of commodity x, $\bar{x}$, is added to the system. The Lagrangian for our Welfare Maximization Problem would come out

$$\mathcal{L} = W + \ldots + \chi[x(P_g) + \bar{x} - \Sigma_i x^i] + \rho[t_{b1}p_{b1} + t_{b2}p_{b2} - B], \text{ so that}$$

$$\frac{\partial\mathcal{L}}{\partial\bar{x}} = \chi > 0, \ \frac{\partial\mathcal{L}}{\partial B} = -\rho < 0, \ \frac{\partial\mathcal{L}}{\partial\bar{x}}d\bar{x} + \frac{\partial\mathcal{L}}{\partial B}dB = 0 \Rightarrow \frac{d\bar{x}}{dB}\Big|_W = -\frac{(\partial\mathcal{L}/\partial B)}{(\partial\mathcal{L}/\partial\bar{x})} = \frac{\rho}{\chi} > 0.$$

Finally note that $B < B^* \Rightarrow \rho < 0$, a case which we disregard.

as the marginal cost, in real terms, of an extra unit of t_{a1}, and

$$\Delta\Pi_b/\Delta t_{a1} \equiv [P_g t_{b1}(\partial g_{b1}/\partial t_{a1}) + g_b(\partial P_g/\partial g)(\Delta g/\Delta t_{a1})]$$

as the marginal impact on bus company's profit[7] from an extra unit of t_{a1}.

The RHS of this expression can be interpreted as follows. The first term, $(P_g t_{b1}(\partial g_{b1}/\partial t_{a1}))$, measures the impact upon congestion (bearing on public firm) of an extra private automobile passenger mile at peak time. This term, therefore, represents a negative <u>technological externality</u> to the bus company since an extra unit of t_{a1} makes more costly, in terms of gasoline needed, the provision of t_{b1} units. On the other hand, $[(\partial P_g/\partial g)(\Delta g/\Delta t_{a1})]$ measures the (partial) increase in the price of gasoline due to an extra private automobile passenger mile at peak time. This increase affects not the requirement of gasoline of the bus firm, but its price and consequently the value of its total consumption of gasoline, g_b. Therefore, $g_b[(\partial P_g/\partial g)(\Delta g/\Delta t_{a1})]$ represents a negative <u>pecuniary externality</u> to our public firm.

Substituting the last two definitions into the expression for τ_{a1}/χ, we get

$$\tau_{a1}/\chi = MC_{a1} + (\rho/\chi)(\Delta\Pi_b/\Delta t_{a1})$$

In words, the social marginal cost - in terms of numeraire - of a t_{a1} unit, τ_{a1}/χ, is equal to the sum of its marginal cost in real terms, MC_{a1}, plus the value of its marginal impact upon bus company's profit, $(\rho/\chi)(\Delta\Pi_b/\Delta t_{a1})$.

The above interpretation generalizes straightforwardly. Thus, defining

(54a) $MC_j \equiv P_g(\Delta g/\Delta t_j)$, $j = a1, a2, b1, b2$

(54b)
$$\Delta\Pi_b/\Delta t_{ak} \equiv P_g t_{bk}(\partial g_{bk}/\partial t_{ak}) + g_b(\partial P_g/\partial g)(\Delta g/\Delta t_{ak}), \quad k = 1,2$$
$$\Delta\Pi_b/\Delta t_{bk} \equiv P_g t_{bk}(\partial g_{bk}/\partial t_{bk}) + g_b(\partial P_g/\partial g)(\Delta g/\Delta t_{bk}) - p_{bk}, \quad k = 1,2$$

expressions (50) through (53) become

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7. See footnote 2 for the definition of Π_b.

$$(55)\quad \tau_{a1}/\chi = MC_{a1} + (\rho/\chi)(\Delta\Pi_b/\Delta t_{a1})$$

$$(56)\quad \tau_{a2}/\chi = MC_{a2} + (\rho/\chi)(\Delta\Pi_b/\Delta t_{a2})$$

$$(57)\quad \tau_{b1}/\chi = MC_{b1} + (\rho/\chi)(\Delta\Pi_b/\Delta t_{b1})$$

$$(58)\quad \tau_{b2}/\chi = MC_{b2} + (\rho/\chi)(\Delta\Pi_b/\Delta t_{b2})$$

Let us substitute equations (55) through (58) into expressions (45), (46) and (48). Then we define

$$(59a)\quad \partial\Pi_b/\partial P_{bk} \equiv [t_{bk} - (\Delta\Pi_b/\Delta t_{a1})(\partial\hat{t}_{a1}/\partial P_{bk}) - (\Delta\Pi_b/\Delta t_{a2})(\partial\hat{t}_{a2}/\partial P_{bk}) - (\Delta\Pi_b/\Delta t_{b1})(\partial\hat{t}_{b1}/\partial P_{bk}) - (\Delta\Pi_b/\Delta t_{b2})(\partial\hat{t}_{b2}/\partial P_{bk})],\ k=1,2$$

$$(59b)\quad \partial\Pi_b/\partial\theta \equiv -[(\Delta\Pi_b/\Delta t_{a1})\Sigma_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + (\Delta\Pi_b/\Delta t_{a2})\Sigma_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} + (\Delta\Pi_b/\Delta t_{b1})\Sigma_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + (\Delta\Pi_b/\Delta t_{b2})\Sigma_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak}],\ k=1,2$$

which can be interpreted, respectively, as the impacts upon bus firm's profit of a small increase in bus fare operating at period k, and of a small increase on the gasoline tax θ. Their signs cannot be predicted a priori.

The resulting expressions can be written in matrix form

$$(60)\quad \begin{bmatrix} \partial\hat{t}_{a1}/\partial P_{b1} & \partial\hat{t}_{a2}/\partial P_{b1} & \partial\hat{t}_{b1}/\partial P_{b1} & \partial\hat{t}_{b2}/\partial P_{b1} \\ \partial\hat{t}_{a1}/\partial P_{b2} & \partial\hat{t}_{a2}/\partial P_{b2} & \partial\hat{t}_{b1}/\partial P_{b2} & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_{k=1}^{2}(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} \end{bmatrix} \begin{bmatrix} MC_{a1}-P_{a1} \\ MC_{a2}-P_{a2} \\ MC_{b1}-P_{b1} \\ MC_{b2}-P_{b2} \end{bmatrix} =$$

$$= \quad (\rho/\chi) \begin{bmatrix} \partial\Pi_b/\partial P_{b1} \\ \partial\Pi_b/\partial P_{b2} \\ \partial\Pi_b/\partial\theta \end{bmatrix}$$

It is now useful to write down the vector of relative divergences between marginal and private costs taking advantage of the set of definitions stated in (54a), and (6) through (9), as

$$(61)\quad \left| \begin{array}{ll} MC_{ak} - P_{ak} = P_g T_{ak} - \theta g_{ak} & \text{with } T_{ak} \equiv t_{ak}(\partial g_{ak}/\partial t_{ak}) + t_{bk}(\partial g_{bk}/\partial t_{ak}) \\ & \qquad k = 1,2 \\ MC_{bk} - P_{bk} = P_g T_{bk} - p_{bk} & \text{with } T_{bk} \equiv t_{bk}(\partial g_{bk}/\partial t_{bk}) + t_{ak}(\partial g_{ak}/\partial t_{bk}) \\ & \qquad k = 1,2 \end{array} \right.$$

where T_{ak}, T_{bk} are pure "intraperiod" congestion effects[8].

With such a break down, the above equation system can be suitably transformed into one which explicitly provides us with the vector of planner's control variables (θ, p_{b1}, p_{b2}), namely

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8. In the next section, we discuss some hypothesis concerning the nature of congestion effects, which are of application here too.

$$(62.1)\quad \begin{bmatrix} \beta_1 & \partial\hat{t}_{b1}/\partial P_{b1} & \partial\hat{t}_{b2}/\partial P_{b1} \\ \beta_2 & \partial\hat{t}_{b1}/\partial P_{b2} & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_{k=1}^{2} \alpha_{ak} g_{ak} & \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ P_{b1} \\ P_{b2} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ A \end{bmatrix}$$

where grouping terms are defined in Appendix 1.1.

The above system refers to the case in which there exists full interdependency of compensated demands across periods and modes. A simpler structure obtains by postulating demand interdependency only across modes, in which case it becomes

$$(62.2)\quad \begin{bmatrix} \beta'_1 & \partial\hat{t}_{b1}/\partial P_{b1} & 0 \\ \beta'_2 & 0 & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_{k} \alpha'_{ak} g_{ak} & \alpha'_{b1} & \alpha'_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ P_{b1} \\ P_{b2} \end{bmatrix} = \begin{bmatrix} B'_1 \\ B'_2 \\ A' \end{bmatrix}$$

where grouping terms are defined in Appendix 1.2.

The simplest structure obtains, of course, when there is no interdependency either across periods or modes. In such a case it becomes

$$(62.3)\quad \begin{bmatrix} 0 & \partial\hat{t}_{b1}/\partial P_{b1} & 0 \\ 0 & 0 & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_{k} \alpha'_{ak} g_{ak} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ P_{b1} \\ P_{b2} \end{bmatrix} = \begin{bmatrix} B''_1 \\ B''_2 \\ A'' \end{bmatrix}$$

and grouping terms are those of Appendix 1.3.

MODEL II. A DIFFERENTIATED BUS FARE STRUCTURE WITHOUT BUDGET CONSTRAINT

We consider now a special case of theoretical interest coming from Model I. We still assume a differentiated structure of prices for bus services, but assumption A7 is deleted, i.e., no budget constraint is considered. (As before the analysis is conducted under three possibilities concerning the structure of demand interdependencies.)

When we suppress constraint (31), conditions (32) and (33) altogether imply $\mu_{b1} = \mu_{b2} = \rho = 0$. With such a result equation (49) now reads

(63) $\gamma/\chi = P_g$

indicating that the marginal social cost of gasoline, g, in terms of numeraire, is equal to its (producer) market price. This result, in turn modifies equations (50) through (53) as follows :

(64) $\dfrac{\tau_{a1}}{\chi} \equiv MC_{a1} = P_g[g_{a1} + t_{a1}(\partial g_{a1}/\partial t_{a1}) + t_{b1}(\partial g_{b1}/\partial t_{a1})]$

(65) $\dfrac{\tau_{a2}}{\chi} \equiv MC_{a2} = P_g[g_{a2} + t_{a2}(\partial g_{a2}/\partial t_{a2}) + t_{b2}(\partial g_{b2}/\partial t_{a2})]$

(66) $\dfrac{\tau_{b1}}{\chi} \equiv MC_{b1} = P_g[g_{b1} + t_{b1}(\partial g_{b1}/\partial t_{b1}) + t_{a1}(\partial g_{a1}/\partial t_{b1})]$

(67) $\dfrac{\tau_{b2}}{\chi} \equiv MC_{b2} = P_g[g_{b2} + t_{b2}(\partial g_{b2}/\partial t_{b2}) + t_{a2}(\partial g_{a2}/\partial t_{b2})]$

By the same token, equations (45) and (46) - taking advantage of (64) through (67) - now read :

(68) $$(MC_{a1} - P_{a1})(\partial\hat{t}_{a1}/\partial P_{b1}) + (MC_{a2} - P_{a2})(\partial\hat{t}_{a2}/\partial P_{b1}) + (MC_{b1} - P_{b1})(\partial\hat{t}_{b1}/\partial P_{b1}) + (MC_{b2} - P_{b2})(\partial\hat{t}_{b2}/\partial P_{b1}) = 0$$

(69) $$(MC_{a1} - P_{a1})(\partial\hat{t}_{a1}/\partial P_{b2}) + (MC_{a2} - P_{a2})(\partial\hat{t}_{a2}/\partial P_{b2}) + (MC_{b1} - P_{b1})(\partial\hat{t}_{b1}/\partial P_{b2}) + (MC_{b2} - P_{b2})(\partial\hat{t}_{b2}/\partial P_{b2}) = 0$$

while equation (48) - the condition on θ - remains unaltered and we reproduce it taking also advantage of equalities (64) through (67) as :

$$(70)\quad (MC_{a1} - P_{a1})\Sigma_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + (MC_{a2} - P_{a2})\Sigma_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} + $$
$$+ (MC_{b1} - P_{b1})\Sigma_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + (MC_{b2} - P_{b2})\Sigma_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} = 0, \; k = 1,2.$$

The set of equations (64) through (70) provides the basic structure of this submodel, to which we now turn to comment. The key point is, of course, from (63), the equality between the marginal social cost of gasoline, in terms of numeraire, and its (producer) price, i.e., $\gamma/\chi = P_g$. This result permits to write down the marginal social cost of each transit commodity, again in terms of numeraire, as being equal to its corresponding marginal cost in real terms. Equations (64) through (67) exemplify that fact. Budget constraint deletion also implies the appearance of zeros on the RHS's of equations (68) and (69). The interpretation of these equations is hopefully straightforward, viz. the sum of relative divergences between social and private marginal costs weighted by Slutsky terms adds up to zero. Equation (70) is reminiscent of (58) and (59) and was commented in the previous section as equation (48). We still refer to it as the condition on θ.

Let us write down the relative divergences between social and private marginal costs taking advantage of equations (64) through (67) and definitions (6) through (9), as follows :

$$(71)\quad (MC_{a1} - P_{a1}) = P_g T_{a1} - \theta g_{a1} \quad \text{with } T_{a1} \equiv t_{a1}(\partial g_{a1}/\partial t_{a1}) + t_{b1}(\partial g_{b1}/\partial t_{a1})$$

$$(72)\quad (MC_{a2} - P_{a2}) = P_g T_{a2} - \theta g_{a2} \quad \text{with } T_{a2} \equiv t_{a2}(\partial g_{a2}/\partial t_{a2}) + t_{b2}(\partial g_{b2}/\partial t_{a2})$$

$$(73)\quad (MC_{b1} - P_{b1}) = P_g T_{b1} - p_{b1} \quad \text{with } T_{b1} \equiv t_{b1}(\partial g_{b1}/\partial t_{b1}) + t_{a1}(\partial g_{a1}/\partial t_{b1})$$

$$(74)\quad (MC_{b2} - P_{b2}) = P_g T_{b2} - p_{b2} \quad \text{with } T_{b2} \equiv t_{b2}(\partial g_{b2}/\partial t_{b2}) + t_{a2}(\partial g_{a2}/\partial t_{b2})$$

Before entering into a deeper discussion about the characterization of solutions for our planner's control variables (θ, p_{b1}, p_{b2}), two remarks are in order. The first one concerns a trivial solution for all systems that will be exploited below. The second one regards some structures of congestion effects T_{ak}, T_{bk} $(k = 1,2)$ that can be considered.

As it will soon be apparent - because of the homogeneity structure of equation systems in the unknowns $(MC_{ak} - P_{ak})$ and $(MC_{bk} - P_{bk})$, $k = 1,2$ - they all admit regardless of the structure of demand interdependencies a trivial first best solution. This comes out by setting equations (71) through (74) equal to zero. It is obvious that in such a case we have

(75a) $\theta = P_g T_{a1}/g_{a1}$

(75b) $\theta = P_g T_{a2}/g_{a2}$

(75c) $p_{b1} = P_g T_{b1}$

(75d) $p_{b2} = P_g T_{b2}$

This solution requires that the RHS's of equalities (75a) and (75b) coincide. Now, since there is no reason to pressume equality between T_{a1}/g_{a1} and T_{a2}/g_{a2} (see below the second remark), and since feasibility conditions prevent the planner from differentiating his tax on gasoline between periods, the solution represented by (75) cannot be considered attainable, and we disregard it in the following.

We now turn to the second remark, which actually concerns three different although related problems, namely :

[R-I] If we assume no congestion in off-peak periods that amounts to write down equations (72) and (74), respectively, as

(76) $(MC_{a2} - P_{a2}) = - \theta g_{a2}$, and

(77) $(MC_{b2} - P_{b2}) = - p_{b2}$,

since $T_{a2} = T_{b2} = 0$, while (71) and (73) remain unaltered.

[R-II] In the case buses do not share the same scarce street space because of the existence of lanes devoted to bus traffic, the use of which is prevented to private motorists, we have at the cost of some ingenuity :

$$(78) \quad (MC_{a1} - P_{a1}) = P_g\hat{T}_{a1} - \theta g_{a1} \quad \text{with} \quad \hat{T}_{a1} \equiv t_{a1}(\partial g_{a1}/\partial t_{a1})$$

$$(79) \quad (MC_{a2} - P_{a2}) = P_g\hat{T}_{a2} - \theta g_{a2} \quad \text{with} \quad \hat{T}_{a2} \equiv t_{a2}(\partial g_{a2}/\partial t_{a2})$$

$$(80) \quad (MC_{b1} - P_{b1}) = P_g\hat{T}_{b1} - p_{b1} \quad \text{with} \quad \hat{T}_{b1} \equiv t_{b1}(\partial g_{b1}/\partial t_{b1})$$

$$(81) \quad (MC_{b2} - P_{b2}) = P_g\hat{T}_{b2} - p_{b2} \quad \text{with} \quad \hat{T}_{b2} \equiv t_{b2}(\partial g_{b2}/\partial t_{b2})$$

In other words, congestion effects across transportation modes have been ruled out.

[R-III] Consider finally the case where public transport has a constant requirement of gasoline per unit of service provided, independent of the time period. (One could think of underground services.) The following version of equations (71) through (74) comes out :

$$(82) \quad (MC_{a1} - P_{a1}) = P_g\hat{T}_{a1} - \theta g_{a1} \quad \text{with} \quad \hat{T}_{a1} = t_{a1}(\partial g_{a1}/\partial t_{a1})$$

$$(83) \quad (MC_{a2} - P_{a2}) = P_g\hat{T}_{a2} - \theta g_{a2} \quad \text{with} \quad \hat{T}_{a2} = t_{a2}(\partial g_{a2}/\partial t_{a2})$$

$$(84) \quad (MC_{b1} - P_{b1}) = - p_{b1} \quad \text{since } T_{b1} = 0$$

$$(85) \quad (MC_{b2} - P_{b2}) = - p_{b2} \quad \text{since } T_{b2} = 0$$

where also $MC_{b1} = MC_{b2} = P_g g_{b1} = P_g g_{b2}$ since $g_{b1} = g_{b2}$.

This last example is an intermediate case of those considered in [R-I] and [R-II], and clearly other combinations are possible. Keeping all this in mind, we turn now to a proper characterization of our planner's control variables (θ, p_{b1}, p_{b2}) in terms of an explicit examination of the structure of (compensated) demand interdependencies.

We begin with the case in which there exists full interdependency across periods and modes. This comes out by writing in matrix form equations (68) through (70) as :

$$(86)\quad \begin{bmatrix} \partial\hat{t}_{a1}/\partial P_{b1} & \partial\hat{t}_{a2}/\partial P_{b1} & \partial\hat{t}_{b1}/\partial P_{b1} & \partial\hat{t}_{b2}/\partial P_{b1} \\ \partial\hat{t}_{a1}/\partial P_{b2} & \partial\hat{t}_{a2}/\partial P_{b2} & \partial\hat{t}_{b1}/\partial P_{b2} & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} \end{bmatrix} \begin{bmatrix} MC_{a1} - P_{a1} \\ MC_{a2} - P_{a2} \\ MC_{b1} - P_{b1} \\ MC_{b2} - P_{b2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This homogeneous system can be appropriately transformed, taking advantage of expressions (71) through (74), into a more appealing system which explicitly gives us the vector of planner's control variables, namely :

$$(87.1)\quad \begin{bmatrix} \beta_1 & \partial\hat{t}_{b1}/\partial P_{b1} & \partial\hat{t}_{b2}/\partial P_{b1} \\ \beta_2 & \partial\hat{t}_{b1}/\partial P_{b2} & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_k \alpha_{ak} g_{ak} & \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = P_g \begin{bmatrix} B_1 \\ B_2 \\ A \end{bmatrix}$$

where grouping terms can be found in Appendix 1.1.

When (compensated) demands are independent between peak and off-peak periods, but not between transportation modes, that is when we have <u>inter-dependency only across modes</u>, the above system modifies as follows :

$$(87.2)\quad \begin{bmatrix} \beta'_1 & \partial\hat{t}_{b1}/\partial P_{b1} & 0 \\ \beta'_2 & 0 & \partial\hat{t}_{b2}/\partial P_{b2} \\ \Sigma\, \alpha'_{ak} g_{ak} & \alpha'_{b1} & \alpha'_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = P_g \begin{bmatrix} B'_1 \\ B'_2 \\ A' \end{bmatrix}$$

and grouping terms are defined in Appendix 1.2.

The simplest case obtains when there is <u>no interdependency either across periods or modes</u>, i.e.

$$(87.3)\quad \begin{bmatrix} 0 & \partial\hat{t}_{b1}/\partial P_{b1} & 0 \\ 0 & 0 & \partial\hat{t}_{b2}/\partial P_{b2} \\ \Sigma\,\alpha'_{ak}\,g_{ak} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = P_g \begin{bmatrix} B''_1 \\ B''_2 \\ A'' \end{bmatrix}$$

and grouping terms are those of Appendix 1.3.

MODEL III. A UNIFORM BUS FARE ACROSS PERIODS WITHOUT BUDGET CONSTRAINT

In the present case we have a model whose basic structure preserves many of the characteristics of Model II. In particular the set of equations (64) through (67) remains unaltered. Equations (68) and (69), however, now admit a unique expression[11], namely

$$(88)\quad (MC_{a1} - P_{a1})(\partial\hat{t}_{a1}/\partial P_b) + (MC_{a2} - P_{a2})(\partial\hat{t}_{a2}/\partial P_b) + (MC_{b1} - P_b)(\partial\hat{t}_{b1}/\partial P_b) + (MC_{b2} - P_b)(\partial\hat{t}_{b2}/\partial P_b) = 0$$

while equation (70) suffers a slight modification (viz. $P_{b1} = P_{b2} = P_b$) and now reads

$$(89)\quad (MC_{a1} - P_{a1})\,\Sigma_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + (MC_{a2} - P_{a2})\,\Sigma_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} + (MC_{b1} - P_b)\,\Sigma_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + (MC_{b2} - P_b)\,\Sigma_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} = 0$$

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11. We are assuming here that our social planner can no longer differentiate bus fares according to peak and off-peak time periods. On the contrary, he is constrained to set a uniform fare between periods. Administrative costs of implementing a differentiated structure of prices and/or political pressures might be reasons that explain this limitation. Planner's problem therefore consists in finding the optimal fare P_b which minimizes misallocation in the transport sector. To that effect the instruments at his disposal are : (i) a tax θ on gasoline purchased by private motorists, (ii) a single fare (both (i) and (ii) being independent of the time period), and (iii) lump sum transfers of income y^i to consumers. In the present case it is also postulated that public firm faces no budget constraint.

    From the viewpoint of bus users, a uniform price means that t_{b1} and t_{b2} are perfect substitutes, that is, total private cost per unit of public transit service on both periods is the same :

    $$P_b = P_{b1} = P_{b2}.$$

    This fact makes a change in the consumer price vector for transit services which now becomes $P = (P_{a1}, P_{a2}, P_b)$ and consequently modifies individual demand functions (17) through (21).

The particular feature of this model is therefore equation (88), and a comment on it is in order.

Equation (88) says, of course, that the sum of relative divergences between social and private marginal costs weighted by (compensated) price effects adds up to zero. A more clear statement, however, can be made by rewriting it as follows

$$\sum_k MC_{ak}(\partial\hat{t}_{ak}/\partial P_b) + \sum_k MC_{bk}(\partial\hat{t}_{bk}/\partial P_b) = \sum_k P_{ak}(\partial\hat{t}_{ak}/\partial P_b) + P_b \sum_k (\partial\hat{t}_{bk}/\partial P_b),$$

which shows that if consumers are to stay on the same indifference curves the sum of marginal costs to society (duly weighted by price effects) of transit services provision must be the same as the sum of individual willingness to pay (again weighted by price effects) for those services. This of course does not necessarily imply equality between marginal costs and prices.

In order to characterize solution values for our planner's control variables (θ, p_{b1}, p_{b2}) we follow the same methodology of previous sections, that is we make three assumptions concerning the structure of (compensated) demand interdependencies. We start with the case of full interdependency across periods[12] and modes by writing (88) and (89) in matrix form as

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11 (cd).

In order to derive equation (88) we replace decision variables P_{b1} and P_{b2} in the original welfare maximization program (22) by P_b. Upon differentiation with respect to P_b taking notice of

a) $\partial V^i/\partial P_b = -\lambda^i(t^i_{b1} + t^i_{b2})$

b) $\partial\hat{t}_j/\partial P_b = \Sigma_i[(t^i_{b1} + t^i_{b2})(\partial t^i_j/\partial y^i) + (\partial t^i_j/\partial P_b)]$ (compensated effects) $j = a1, a2, b1, b2$

$\partial\hat{x}/\partial P_b = \Sigma_i[(t^i_{b1} + t^i_{b2})(\partial x^i/\partial y^i) + (\partial x^i/\partial P_b)]$

c) (64) through (67),

we get our desired expression.

12. It should be noted that our assumption of a uniform bus fare between time periods makes of necessity $\partial\hat{t}_j/\partial P_b$, $j = b1, b2$, different from zero. This fact makes misleading the heading "no interdependency either across periods or modes" of system (93.3). The same remark applies to (98.3) in Model IV.

$$(90)\quad \begin{bmatrix} \partial\hat{t}_{a1}/\partial P_b & \partial\hat{t}_{a2}/\partial P_b & \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \sum_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} \end{bmatrix} \begin{bmatrix} MC_{a1} - P_{a1} \\ MC_{a2} - P_{a2} \\ MC_{b1} - P_b \\ MC_{b2} - P_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As before we transform the equation system (90), taking notice of expressions (71) through (74), into a system which explicitly provides us with the vector of planner's control variables, that is

$$(91)\quad \begin{bmatrix} \beta & \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \sum_k \alpha_{ak} g_{ak} & \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} C \\ A \end{bmatrix} P_g$$

where grouping terms are defined in Appendix 1.4.

Contrary to Models I and II we have now a two system equations in three unknowns, so that we have a degree of freedom. Some insight can be gained if we restate (91) as

$$(92.1)\quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} P_g C - \beta\theta \\ P_g A - \Sigma_k \alpha_{ak} g_{ak} \theta \end{bmatrix}$$

With such a formulation it is clear that all we can obtain are tolls $p_{b1}(\theta)$ and $p_{b2}(\theta)$, where θ is now an exogenous variable[13].

When there exists <u>interdependency only across modes</u>, the above system admits the following form :

13. Obviously we could have also chosen either p_{b1} or p_{b2} as the exogenous variable.

$$(92.2) \quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \alpha'_{b1} & \alpha'_{b2} \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} P_g C - \beta\theta \\ P_g A' - \Sigma_k \alpha'_{ak} g_{ak} \theta \end{bmatrix}$$

where grouping terms can be found in Appendix 1.4.

Finally, when there exists no interdependency either across periods or modes, we obtain the following system[14]

$$(93.3) \quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} P_g C' \\ P_g A'' - \Sigma \alpha'_{ak} g_{ak} \theta \end{bmatrix}$$

where grouping terms are defined in Appendix 1.4.

14. Straightforward calculation shows that

$$\theta = P_g A'' / \Sigma_k \alpha'_{ak} g_{ak}, \text{ and}$$

$$p_{b1} = [P_g C' - (\partial\hat{t}_{b2}/\partial P_b) p_{b2}] / (\partial\hat{t}_{b1}/\partial P_b)$$

Notice that in this subcase p_{b1} and p_{b2} do not depend on θ.

MODEL IV. A UNIFORM BUS FARE WITH A BUDGET CONSTRAINT

In this case we have a model whose basic structure is quite similar to Model I. More specifically, the set of equations (50) through (53) remains the same. Equations (45) and (46), however, now admit a unique expression[15], i.e.,

$$(93)\quad [(\tau_{a1}/\chi) - P_{a1}](\partial\hat{t}_{a1}/\partial P_b) + [(\tau_{a2}/\chi) - P_{a2}](\partial\hat{t}_{a2}/\partial P_b) + $$
$$+ [(\tau_{b1}/\chi) - P_b](\partial\hat{t}_{b1}/\partial P_b) + [(\tau_{b2}/\chi) - P_b](\partial\hat{t}_{b2}/\partial P_b) = (\rho/\chi)(t_{b1} + t_{b2})$$

while equation (48) suffers a slight modification (viz. $P_{b1} = P_{b2} = P_b$) and now reads

$$(94)\quad [(\tau_{a1}/\chi) - P_{a1}]\Sigma_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + [(\tau_{a2}/\chi) - P_{a2}]\Sigma_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} +$$
$$+ [(\tau_{b1}/\chi) - P_b]\Sigma_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + [(\tau_{b2}/\chi) - P_b]\Sigma_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} = 0,\ k=1,2$$

The basic feature of this model, as in the previous one, is equation (93) which admits the following comment : The sum of divergences between social and private marginal costs weighted by pure price effects adds up to bus firm's (global) supply of transit services, $(t_{b1} + t_{b2})$, times the shadow price of an extra unit of budget requirement B.

Taking advantage of definitions (55) through (58) and (59b), and defining

$$(95)\quad \partial\Pi_b/\partial P_b \equiv [t_{b1} + t_{b2} - (\Delta\Pi_b/\Delta t_{a1})(\partial\hat{t}_{a1}/\partial P_b) - (\Delta\Pi_b/\Delta t_{a2})(\partial\hat{t}_{a2}/\partial P_b)$$
$$- (\Delta\Pi_b/\Delta t_{b1})(\partial\hat{t}_{b1}/\partial P_b) - (\Delta\Pi_b/\Delta t_{b2})(\partial\hat{t}_{b2}/\partial P_b)]$$

as the impact upon public firm's profit of a small change in uniform bus fare P_b.

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15. Equation (93) can be obtained following steps analogous to those described in footnote 11. The analogy is not complete because of the existence of the budget constraint.

equations (93) and (94) can be stated in matrix form as :

$$(96) \quad \begin{bmatrix} \partial\hat{t}_{a1}/\partial P_b & \partial\hat{t}_{a2}/\partial P_b & \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \sum_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} & \sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} \end{bmatrix} \begin{bmatrix} MC_{a1} - P_{a1} \\ MC_{a2} - P_{a2} \\ MC_{b1} - P_b \\ MC_{b2} - P_b \end{bmatrix} =$$

$$= (\rho/\chi) \begin{bmatrix} \partial\Pi_b/\partial P_b \\ \partial\Pi_b/\partial\theta \end{bmatrix}$$

The equation system (96) can be transformed into one which yields the vector of planner's control variables (θ, p_{b1}, p_{b2}) after substitution (with a minor qualification[16]) for the set of definitions in (61). The resulting system is

$$(97) \quad \begin{bmatrix} \beta & \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \Sigma\,\alpha_{ak}\,g_{ak} & \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} C \\ A \end{bmatrix}$$

where grouping terms can be found in Appendix 1.5.

Choosing θ as the exogenous variable (97) can be rewritten as

$$(98.1) \quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} C - \beta\theta \\ A - \Sigma_k\,\alpha_{ak}g_{ak}\theta \end{bmatrix}$$

where grouping terms can be found in Appendix 1.5

16. In effect, the expression for ($MC_{bk} - P_{bk}$) in (61) now reads ($MC_{bk} - P_b$), the RHS being unchanged.

The above discussion refers to the case in which there exists full interdependency of (compensated) demands across periods and modes. When we postulate interdependency only across modes, a simpler structure obtains for the above system, namely :

$$(98.2)\quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ \alpha'_{b1} & \alpha'_{b2} \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} C - \beta\theta \\ A' - \Sigma_k \alpha'_{ak} g_{ak}\theta \end{bmatrix}$$

where grouping terms are those of Appendix 1.5

Finally, when there exists no interdependency either across periods or modes, the simplest system obtains, i.e.,

$$(98.3)\quad \begin{bmatrix} \partial\hat{t}_{b1}/\partial P_b & \partial\hat{t}_{b2}/\partial P_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} C' \\ A'' - \Sigma\alpha'_{ak} g_{ak}\theta \end{bmatrix}$$

where grouping terms are defined in Appendix 1.5. Note that in this last subcase p_{b1} and p_{b2} are independent of θ.

MODEL V. A DIFFERENTIATED BUS FARE STRUCTURE WITH BUDGET CONSTRAINT AND A UNIFORM LUMP SUM TRANSFER

In this section we finally extend Model I to the case in which lump sum transfers of income y^i to consumers are no longer an instrument available to our social planner. Instead we postulate that he/she is constrained to implement a uniform lump sum transfer, y.

With such a limitation, the following modifications are in order. Budget constraints now read

(99) $$P_{a1}t^i_{a1} + P_{a2}t^i_{a2} + P_{b1}t^i_{b1} + P_{b2}t^i_{b2} + x^i = y \quad \forall i,\ i = 1,\ldots,n.$$

Maximization of (1) subject to (99) yields demand functions

(100) $$t^i_j = t^i_j(P,y), \quad i = 1,\ldots,n;\ j = a1,\ a2,\ b1,\ b2$$
$$x^i = x^i(P,y), \quad i = 1,\ldots,n.$$

The Welfare Maximization Program becomes

(101) $$\underset{\substack{(p_{b1},p_{b2},P_{b1},P_{b2},P_g,\theta,\\ y,t_{a1},t_{a2},t_{b1},t_{b2})}}{\mathrm{Max}} \quad W \equiv \sum_i \beta^i U^i[t^i_{a1}(P,y),t^i_{a2}(P,y),t^i_{b1}(P,y),t^i_{b2}(P,y),x^i(P,y)]$$

subject to (23) through (31) as before. Notice that decision variable y has been substituted for decision variables y^i in the original formulation. Also notice that these constraints imply

(102) $$Y \equiv ny = x + P_g\, g + \theta(g_{a1}t_{a1} + g_{a2}t_{a2}) + p_{b1}t_{a1} + p_{b2}t_{b2}.$$

In words, aggregate income, Y, consisting of the proceeds from gasoline tax on private users and the sum of "tolls" collected, is redistributed to the n consumers according to the uniform lump sum transfer y.

In addition to (23) through (31), first order conditions (32) through (42) remain unaltered except (38) that now reads

$$(103) \quad \sum_i \beta^i\lambda^i - \tau_{a1}\sum_i (\partial t^i_{a1}/\partial y) - \tau_{a2}\sum_i (\partial t^i_{a2}/\partial y) - \tau_{b1}\sum_i (\partial t^i_{b1}/\partial y) - \tau_{b2}\sum_i (\partial t^i_{b2}/\partial y) - \chi\sum_i (\partial x^i/\partial y) = 0.$$

We cannot directly substitute (103) into (34) through (37) as before, so that a more laborious procedure must be followed.

Let us begin with a definition :

$$(104) \quad \alpha^i \equiv \beta^i\lambda^i$$

where α^i stands for the social evaluation of the increased utility of i made possible by higher income, and it is the product of the (private) marginal utility of income for consumer i, λ^i, and the social weight accorded to his utility, β^i.

We first use Slutsky equations

$$(105) \quad \begin{aligned} \partial t^i_h/\partial P_k &= \partial \hat{t}^i_h/\partial P_k - (\partial t^i_h/\partial y)t^i_k \qquad h,k = a1,a2,b1,b2 \\ \partial x^i/\partial P_k &= \partial \hat{x}^i/\partial P_k - (\partial x^i/\partial y)t^i_k \qquad k = a1,a2,b1,b2 \end{aligned}$$

and the above definition of α^i to substitute into first order conditions (34) through (37). Then we define :

$$(106) \quad v^i \equiv \alpha^i - \tau_{a1}(\partial t^i_{a1}/\partial y) - \tau_{a2}(\partial t^i_{a2}/\partial y) - \tau_{b1}(\partial t^i_{b1}/\partial y) - \tau_{b2}(\partial t^i_{b2}/\partial y) - \chi(\partial x^i/\partial y)$$

whose interpretation is given below, and note that (103) implies $\Sigma_i v^i = 0$. Finally, we further substitute (106) into the resulting expressions for (34) through (37), taking advantage of (32) and (33) and of the following fact upon compensated demand effects

$$(107) \quad P_{a1}\frac{\partial \hat{t}^i_{a1}}{\partial P_k} + P_{a2}\frac{\partial \hat{t}^i_{a2}}{\partial P_k} + P_{b1}\frac{\partial \hat{t}^i_{b1}}{\partial P_k} + P_{b2}\frac{\partial \hat{t}^i_{b2}}{\partial P_k} + \frac{\partial \hat{x}^i}{\partial P_k} = 0, \ k = a1,a2,b1,b2.$$

With this manipulation we get the corresponding expressions for equations (45) through (48), i.e.,

$$(108)\quad [(\tau_{a1}/\chi) - P_{a1}](\partial\hat{t}_{a1}/\partial P_{b1}) + [(\tau_{a2}/\chi) - P_{a2}](\partial\hat{t}_{a2}/\partial P_{b1}) +$$

$$+ [(\tau_{b1}/\chi) - P_{b1}](\partial\hat{t}_{b1}/\partial P_{b1}) + [(\tau_{b2}/\chi) - P_{b2}](\partial\hat{t}_{b2}/\partial P_{b1}) =$$

$$= (1/\chi)(\rho t_{b1} - \Sigma_i v^i t^i_{b1})$$

$$(109)\quad [(\tau_{a1}/\chi) - P_{a1}](\partial\hat{t}_{a1}/\partial P_{b2}) + [(\tau_{a2}/\chi) - P_{a2}](\partial\hat{t}_{a2}/\partial P_{b2}) +$$

$$+ [(\tau_{b1}/\chi) - P_{b1}](\partial\hat{t}_{b1}/\partial P_{b2}) + [(\tau_{b2}/\chi) - P_{b2}](\partial\hat{t}_{b2}/\partial P_{b2}) =$$

$$= (1/\chi)(\rho t_{b2} - \Sigma_i v^i t^i_{b2}).$$

$$(110)\quad [(\tau_{a1}/\chi) - P_{a1}]\sum_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + [(\tau_{a2}/\chi) - P_{a2}]\sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} +$$

$$+ [(\tau_{b1}/\chi) - P_{b1}]\sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + [(\tau_{b2}/\chi) - P_{b2}]\sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} =$$

$$= (1/\chi)[-\sum_i(t^i_{a1}g_{a1} + t^i_{a2}g_{a2})v^i - \rho(t_{b1}g_{b1} + t_{b2}g_{b2}) + \gamma(\partial g/\partial P_g) + \chi(\partial x/\partial P_g)]$$

$$(111)\quad [(\tau_{a1}/\chi) - P_{a1}]\sum_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} + [(\tau_{a2}/\chi) - P_{a2}]\sum_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} +$$

$$+ [(\tau_{b1}/\chi) - P_{b1}]\sum_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} + [(\tau_{b2}/\chi) - P_{b2}]\sum_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} =$$

$$= (1/\chi)[-\sum_i(t^i_{a1}g_{a1} + t^i_{a2}g_{a2})v^i]$$

On direct inspection equations (110) and (111) clearly show that (49) is verified. This in turn means that the block of equations (50) through (53) remains unaltered. Consequently, the basic structure of the present model consists, in addition to that block, of equations (108), (109) and (111).

Let us interpret now the new features of the model under consideration coming out in equations (108), (109) and (111). These are the terms in v^i appearing on the RHS's, and its economic meaning may be seen from looking e.g. at the right hand side of the first of these equations. We have

$$(1/\chi)(\rho t_{b1} - \sum_i v^i t^i_{b1}).$$

The first term is already familiar to us. To interpret the term $\Sigma_i v^i t^i_{b1}$, let firstly define

$$\overline{t}_{b1} \equiv (1/n) \sum_i t^i_{b1} = t_{b1}/n$$

and

$$\overline{v} \equiv (1/n) \sum_i v^i = 0 \quad \text{(by 103)}.$$

Then, we have

$$\sum_i v^i t^i_{b1} = \sum_i (v^i - \overline{v})(t^i_{b1} - \overline{t}_{b1}) = \sum_i v^i (t^i_{b1} - \overline{t}_{b1}).$$

Following Feldstein (1973), we could refer to $\Sigma_i v^i t^i_{b1}$ as the <u>distributional characteristic</u> of commodity t_{b1}. The distributional characteristic is then the covariance of the social marginal (utility, efficiency) value of income and of the consumption of commodity t_{b1}.

Distributional considerations appear, therefore, in pricing formulas through the additional term $\Sigma_i v^i \ldots$ In order to take optimal decisions, our bus company will have to know not only the distributional characteristics of the transit commodities that it produces, but also those of t_{a1} and t_{a2}.

It is important to notice the fact that all terms in v^i enter additively on the RHS's. This means that all intermediate steps undertaken in Model I are of application here too, so that we end up with the following final equations :

(112) $$(MC_{a1} - P_{a1})(\partial\hat{t}_{a1}/\partial P_{b1}) + (MC_{a2} - P_{a2})(\partial\hat{t}_{a2}/\partial P_{b1}) + (MC_{b1} - P_{b1})(\partial\hat{t}_{b1}/\partial P_{b1})$$
$$+ (MC_{b2} - P_{b2})(\partial\hat{t}_{b2}/\partial P_{b1}) = (1/\chi)[\rho(\partial\Pi_b/\partial P_{b1}) - \sum_i v^i (t^i_{b1} - \overline{t}_{b1})]$$

with $\overline{t}_{b1} = t_{b1}/n$

(113) $$(MC_{a1} - P_{a1})(\partial\hat{t}_{a1}/\partial P_{b2}) + (MC_{a2} - P_{a2})(\partial\hat{t}_{a2}/\partial P_{b2}) + (MC_{b1} - P_{b1})(\partial\hat{t}_{b1}/\partial P_{b2})$$
$$+ (MC_{b2} - P_{b2})(\partial\hat{t}_{b2}/\partial P_{b2}) = (1/\chi)[\rho(\partial\Pi_b/\partial P_{b2}) - \sum_i v^i (t^i_{b2} - \overline{t}_{b2})]$$

with $\overline{t}_{b2} = t_{b2}/n$

$$(114) \quad (MC_{a1} - P_{a1}) \sum_k (\partial\hat{t}_{a1}/\partial P_{ak}) g_{ak} + (MC_{a2} - P_{a2}) \sum_k (\partial\hat{t}_{a2}/\partial P_{ak}) g_{ak} +$$

$$+ (MC_{b1} - P_{b1}) \sum_k (\partial\hat{t}_{b1}/\partial P_{ak}) g_{ak} + (MC_{b2} - P_{b2}) \sum_k (\partial\hat{t}_{b2}/\partial P_{ak}) g_{ak} =$$

$$= (1/\chi)[\rho(\partial\Pi_b/\partial\theta) - \sum_i v^i \sum_k (t^i_{ak} - \bar{t}_{ak}) g_{ak}] \quad \text{with } \bar{t}_{ak} = t_{ak}/n, \ k = 1,2.$$

Following the same steps that in Model I, the system of equations represented by (112) through (114) can be transformed into one yielding explicitly the vector of planner's control variables (θ, p_{b1}, p_{b2}). In the present case it would be

$$(115) \quad \begin{bmatrix} \beta_1 & \partial\hat{t}_{b1}/\partial P_{b1} & \partial\hat{t}_{b2}/\partial P_{b1} \\ \beta_2 & \partial\hat{t}_{b1}/\partial P_{b2} & \partial\hat{t}_{b2}/\partial P_{b2} \\ \sum_k \alpha_{ak} g_{ak} & \alpha_{b1} & \alpha_{b2} \end{bmatrix} \begin{bmatrix} \theta \\ p_{b1} \\ p_{b2} \end{bmatrix} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{A} \end{bmatrix}$$

where
$$\hat{B}_1 \equiv P_g B_1 - (\rho/\chi)(\partial\Pi_b/\partial P_{b1}) + (1/\chi) \sum_i v^i (t^i_{b1} - \bar{t}_{b1})$$

$$\hat{B}_2 \equiv P_g B_2 - (\rho/\chi)(\partial\Pi_b/\partial P_{b2}) + (1/\chi) \sum_i v^i (t^i_{b2} - \bar{t}_{b2})$$

$$\hat{A} \equiv P_g A - (\rho/\chi)(\partial\Pi_b/\partial\theta) + (1/\chi) \sum_i v^i \sum_k (t^i_{ak} - \bar{t}_{ak}) g_{ak}$$

and grouping terms are those of Appendix 1.1.

The above system refers to the case in which there exists <u>full interdependency across periods and modes</u>. Comparing (115) with (62.1), we see that they only differ in the terms in $\Sigma_i v^i \ldots$ that enter additively on the RHS. That suggests that extension to other structures of demand interdependencies is straightforward and we do not undertake it here.

APPENDIX 1.1

$\alpha_{a1} = \Sigma_k(\partial\hat{t}_{a1}/\partial P_{ak})g_{ak} \qquad k = 1,2$

$\alpha_{a2} = \Sigma_k(\partial\hat{t}_{a2}/\partial P_{ak})g_{ak} \qquad k = 1,2$

$\alpha_{b1} = \Sigma_k(\partial\hat{t}_{b1}/\partial P_{ak})g_{ak} \qquad k = 1,2$

$\alpha_{b2} = \Sigma_k(\partial\hat{t}_{b2}/\partial P_{ak})g_{ak} \qquad k = 1,2$

$\beta_1 = \Sigma_k(\partial\hat{t}_{ak}/\partial P_{b1})g_{ak} \qquad k = 1,2$

$\beta_2 = \Sigma_k(\partial\hat{t}_{ak}/\partial P_{b2})g_{ak} \qquad k = 1,2$

$B_k = \Sigma_j(\partial\hat{t}_j/\partial P_{bk})T_j \qquad j = a1,a2,b1,b2 \ ; \ k = 1,2$

$A = \Sigma_j\alpha_j T_j \qquad j = a1,a2,b1,b2$

$\mathcal{B}_k = P_g B_k - (\rho/\chi)(\partial\Pi_b/\partial P_{bk}) \quad k = 1,2 \ ; \ \partial\Pi_b/\partial P_{bk}$ defined in (59a)

$\mathcal{A} = P_g A - (\rho/\chi)(\partial\Pi_b/\partial\theta)$

APPENDIX 1.2

$$\alpha'_{ak} = (\partial\hat{t}_{ak}/\partial P_{ak})g_{ak} \qquad k = 1,2$$

$$\alpha'_{bk} = (\partial\hat{t}_{bk}/\partial P_{ak})g_{ak} \qquad k = 1,2$$

$$\beta'_{k} = (\partial\hat{t}_{ak}/\partial P_{bk})g_{ak} \qquad k = 1,2$$

$$B'_{k} = (\partial\hat{t}_{ak}/\partial P_{bk})T_{ak} + (\partial\hat{t}_{bk}/\partial P_{bk})T_{bk} \qquad k = 1,2$$

$$A' = \Sigma_j \alpha'_j T_j \qquad j = a1,a2,b1,b2$$

$$\mathcal{B}'_{k} = P_g B'_k - (\rho/\chi)(\partial\Pi'_b/\partial P_{bk}) \qquad k = 1,2$$

$$\mathcal{A}' = P_g A' - (\rho/\chi)(\partial\Pi'_b/\partial\theta)$$

$$(\partial\Pi'_b/\partial P_{bk}) = \left[t_{bk} - (\Delta\Pi_b/\Delta t_{ak})(\partial\hat{t}_{ak}/\partial P_{bk}) - (\Delta\Pi_b/\Delta t_{bk})(\partial\hat{t}_{bk}/\partial P_{bk})\right] \qquad k=1,2$$

$$(\partial\Pi'_b/\partial\theta) = -\Sigma_j(\Delta\Pi_b/\Delta t_j)\alpha'_j \qquad j = a1,a2,b1,b2$$

APPENDIX 1.3

$$B''_k = (\partial \hat{t}_{bk}/\partial P_{bk})T_{bk} \qquad k = 1,2$$

$$A'' = \Sigma_j \alpha'_j T_j \qquad j = 1,2$$

$$\mathcal{B}''_k = P_g B''_k - (\rho/\chi)(\partial \Pi''_b/\partial P_{bk}) \qquad k = 1,2$$

$$\mathcal{A}'' = P_g A'' - (\rho/\chi)(\partial \Pi''_b/\partial \theta).$$

$$\partial \Pi''_b/\partial P_{bk} = [t_{bk} - (\Delta \Pi_b/\Delta t_{bk})(\partial \hat{t}_{bk}/\partial P_{bk})] \qquad k = 1,2$$

$$\partial \Pi''_b/\partial \theta = - \Sigma_j (\Delta \Pi_b/\Delta t_j)\alpha'_j \qquad j = a1,a2$$

APPENDIX 1.4

$$\beta = \Sigma_k (\partial \hat{t}_{ak} / \partial P_b) g_{ak}$$

$$C = \Sigma_j (\partial \hat{t}_j \, / \partial P_b) T_j \qquad j = a1, a2, b1, b2$$

$$C' = \Sigma_k (\partial \hat{t}_{bk} / \partial P_b) T_{bk} \qquad k = 1,2$$

All other grouping terms are defined as in previous appendices

APPENDIX 1.5

$$C = P_g C - (\rho/\chi)(\partial \Pi_b / \partial P_b)$$

$$C' = P_g C' - (\rho/\chi)(\partial \Pi_b' / \partial P_b)$$

$$(\partial \Pi_b' / \partial P_b) = [t_{b1} + t_{b2} - (\Delta \Pi_b / \Delta t_{b1})(\partial \hat{t}_{b1} / \partial P_b) - (\Delta \Pi_b / \Delta t_{b2})(\partial \hat{t}_{b2} / \partial P_b)]$$

All other grouping terms as in previous appendices

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