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SOME REFLECTIONS ON OPTIMAL RULES FOR PUBLIC INVESTMENT UNDER UNCERTAINTY

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1982

Candidat a la convocatòria de [illegible]
1983/84

1. INTRODUCTION

It is a fact that uncertainty imposes risk taking behaviour to private investors. Accepting the value judgement that the State is only the collection of individuals comprising society, is it appropriate to discount public investment in the same way as private investment, or not ?

This is a debated issue, the nature of which depends on the financial arrangements society has at its disposal. Sections 2 through 4 summarize the most well known propositions on the subject. Section 5 explores in depth the two last contributions to the debate.

2. PUBLIC INVESTMENT IN THE ARROW-DEBREU FRAMEWORK

The most well developed theory of resource allocation under uncertainty is no doubt the Arrow-Debreu state-contingent commodity approach[1]. In essence, this approach converts the economy into an expanded system of insurance markets[2] in each of which there is a price known with certainty. The standard analysis of consumption and production choices under certainty applies directly to the state-contingent commodities, and hence known theorems can be applied.

The implications of this theory for public sector investment have been explored by Hirshleifer. His conclusion is clear:

> "The efficient discount rate, assuming perfect markets, is the market rate implicit in the valuation of private assets whose returns are "comparable" to the public investment in question-where "comparable" means having the same proportionate time-state distribution of returns".

Hirshleifer also shows that in general the rate of discount on public sector investments should contain a risk premium, though it will vary with the project, and may be positive or negative. It is calculated from the prices for the state-contingent outputs which the investment will generate. Arrow-Lind, however, have proved a theorem which establishes conditions under which a riskless public discount rate should be used[3].

The Arrow-Debreu economy, however, presents a fundamental difficulty, which is that we cannot assume the existence in fact of state-contingent commodities and their markets[4], and so the model loses some of its normative interest.

3. IMPERFECT SOLUTIONS TO IMPERFECT MARKETS : RISK POOLING VS. RISK SPREADING

In view of the defficiencies that present capital markets show relative to the perfect insurance markets constructed by Arrow and Debreu, some authors have pointed out two reasons why the government may not follow the risk-taking behaviour of private investors when evaluating investment projects. These are the phenomena of risk pooling and risk spreading to which we turn now to comment.

3.1 Risk pooling: the Samuelson and Vickrey view.

Without going into details, their position can be summarized as follows. Individual enterprises are too small to be willing to undertake socially desirable risk investment. The high rates of return found on investments in the private sector are the result of discounting for risk, and differences in rates of return across alternative lines of investment reflect differences in pooling[5] opportunities open to investors. The government, however, invests in a great number of divers projects and is able to pool risk to a much greater extent than private investors. Therefore, the discount rate appropriate for evaluating public investments should be lower than the discount rate used to evaluate comparable investments in the private sector. Moreover, the extremely large and diversified investment portfolio held by the public sector, justifies the government in using the expected present value of a project as an approximate measure of its contribution to social welfare without taking risk into account.

3.2 Risk spreading: the Arrow-Lind proposition.

Risk spreading occurs when the benefits and costs of a project are shared among a large number of individuals. In the case of public projects financed out of general tax revenues risk-spreading takes place among individual taxpayers. As Arrow-Lind have argued:

AL.1 if all the benefits and costs of a public sector investment are recovered or borne by the government, which then determines its total taxation in the light of them.

AL.2 if these benefits are uncorrelated with national income, and

AL.3 if the project is small relative to national income;

then, the total social cost of risk bearing associated with any individual investment tends to zero as the number of taxpayers tends to infinity, and so for large populations, it can effectively be ignored. Hence, in this case the discount rate would not contain a risk premium.

All three conditions stated above are necessary. If the recipients of the outputs of the investment are risk-averse and retain net benefits, then these should be discounted for risk. If the net benefits are correlated with national income, a risk premium on the discount rate will be appropriate, and its sign will vary directly with the sign of the correlation in question. Finally, the argument is also inapplicable for a project with a sufficiently large variance to affect individuals significantly.

3.3 Risk pooling vs. Risk spreading: the James proposition.

Risk spreading applies in the many-investor case, regardless of the number of projects. Risk pooling applies in the many-project case, regardless of the number of investors. The former is an argument for piecemeal evaluation of projects, the latter for their global evaluation. James has pointed out that there may be an inconsistency between these two arguments and concludes that:

"Piecemeal decisions based on risk spreading will lead to a correct global outcome if and only if sufficiently large gains from risk pooling are also present, when the investments are reevaluated as a group."

3.4 Evaluation.

The relevance, in isolation, of the Arrow-Lind proposition to public sector resource allocation is very much in doubt. In mixed economies, the returns to most public enterprise investments are highly correlated with national income (it should be said that Arrow-Lind appear to have in mind small localised investments in non-marketed outputs, or at least those which have strong public good elements[6]), while net benefits of public goods are certainly not recovered.

The very presence of highly correlated projects (with national income) invalidates also the argument for zero risk premium in the discount rates of public projects, which is the extreme conclusion reached by Samuelson and Vickrey. To see what is involved we can refer to a contribution by Jensen and long in a CAPM framework. According to them, necessary conditions for the simultaneous existence of zero social risk and positive private risk are:

BJ.1 the existence of zero covariance among the returns on a very large (strictly infinite) number of assets, and

BJ.2 the nonexistance of perfect markets for fractional claims on the returns of assets.

The following passage deserves quoting:

BJ.3 "As long as there are perfect markets for fractional claims on the returns of assets, individuals in the private sector can through diversification (pooling), reduce their private risk to the average covariance among assets, which is an irreducible social risk... In view of the evidence that virtually all assets sold in equity markets have returns that are positively intercorrelated, it is obvious that neither perfectly functioning risk markets nor the total transfer of private sector activities to the public sector will eliminate this risk."

To this one can add that in the purely private economy contemplated in the basic CAPM, risk is also shared efficiently among individual shareholders in the economy. The important point to note here is that these critics come from an analysis where the perfect insurance markets of Arrow and Debreu have been replaced by a perfect stock market[7]. The corollary is that the only barriers to a socially optimal risk level are imperfections in the stock market. Is the Hirshleifer prescription still mandatory in this framework ? To answer this question we devote the following sections.

4. OPTIMAL INVESTMENT RULES FOR PUBLIC INVESTMENT IN A MIXED ECONOMY: SANDMO'S CONTRIBUTION

Sandmo gives the most robust defense and limitation of the Hirshleifer view for a particular mixed economy. He considers a mixed economy in which any activity undertaken by the public sector is replicated in the private sector. More technically, he makes a partition of all economic activities into industries according to the Modigliani-Miller concept of risk classes. In industry j, say, the ratio X_j / X_j^* of private and public production is independent of the state of the world. Otherwise said, outputs in the private and public part of each industry are perfectly correlated.

He then examines two structures of private capital markets. In the first one, where all private firms are organised as corporations and their shares are traded in a perfect stock market, he finds mandatory the Hirshleifer prescription for public investment. That is, " risk margins " in the private sector represent a social evaluation of the risk associated with each type of investment, and public investment must imitate private investment. Market data contain all the relevant information.

In the second framework, where there is no stock market, no such a clear cut prescription can be drawn.

The critics to be addressed to this contribution are essentialy the same ones that Mossin directs to Diamond, whose work provides the basis for Sandmo's contribution. These are two, namely:

a) The case of de_composable production functions is a very special one and hence of limited interest. It excludes from consideration stochastic dependence among output of different firms; and

b) The valuation model utilised, based on the proportionality assumption[8], is generally not valid.

5. OPTIMAL INVESTMENT RULES FOR PUBLIC INVESTMENT IN A MIXED ECONOMY UNDER A MODIFIED SLM MODEL

5.1 Abstract

The last two contributions to the debate have emerged under the framework of a modified SLM model. This model permits to calculate equilibrium values for private firms with arbitrary output patterns and allows explicit consideration of stochastic dependence among different firms. This model is presented in section 5.2. In section 5.3 the formulations of Stapleton and Subrahmanyam (1978) and Holmstrom are summarized in the two investors-two firms framework they use. Section 5.4 presents Stapleton's-Subrahmanyam's optimal investment rules. Finally, in section 5.5 we present a reformulated version of Holmstrom's work.

Issue: the very presence of public firms induces an imperfect distribution of risk in an otherwise perfect stock market economy. This dictates a higher discount rate for a firm after its nationalization relative to the one it would use in a purely private economy (Stap. and Sub.). In an already existing mixed economy, if a public and a private firm face the same opportunity to invest in a new project no unambiguos comparison of costs of capital seems possible (Holmstrom):

5.2 Valuation model in a mixed economy[9].

1. The Setting and Assumptions

We consider an economy consisting of H individuals (= investors = taxpayers) and n firms. The following assumptions are made:

V.1 Single period framework.

V.2 Firms. Private firms $j = 1,\ldots,\ell$ (or $j \in P$) generate random returns $\tilde{X}_j$ which are distributed to investors according to their perfectly tradable share proportions Z_{ij}, $i \in H$, $j \in P$. Public firms $j = \ell + 1,\ldots,n$ (or $j \in N$) generate random returns $\tilde{X}_j$ which are arbitrarily allocated through the tax system[10] to the individuals in the economy in nontradable proportions α_{ij}, $i \in H$, $j \in N$. All $\tilde{X}_j$, $j=1,\ldots,n$, are normally distributed.

V.3 Investors

(i) They are risk averse and maximize CARA[11] utility functions over their final uncertain wealth. This assumption together with the normal distribution of the $\tilde{X}_j$, $j = 1,\ldots,n$ permits to write their utility functions in terms of the mean and variance of their final wealth: $U_i = f_i\ (E_i, V_i)$, $i \in H$

(ii) They have present wealth, W_i, in the form of bonds, $\bar{m}_i$, and share endowments of private firms, $\bar{Z}_{ij}$, $i \in H$, $j \in P$. As taxpayers their future wealth will be affected by their holdings α_{ij} in the public firm j, $i \in H$, $j \in N$.

(iii) They have homogeneus expectations with respect to the means, variances and covariances of all random returns in the economy. The following notation will be used:

$$
(1)\quad \left[\begin{array}{lll|lll}
\mu_j = E(\tilde{X}_j) & & j = 1,\ldots,n & B = [\sigma_{jk}] & & j,k \in P \\
\sigma_{jj} = V(\tilde{X}_j) & & j = 1,\ldots,n & C = [\sigma_{jk}] & & j,k \in N \\
\sigma_{jk} = Cov(\tilde{X}_j, \tilde{X}_k) & & j,k = 1,\ldots,n & D = [\sigma_{j,k}] & & j(k) \in P,\ k(j) \in N \\
A = [\sigma_{jk}] & & j,k = 1,\ldots,n & & & \\
\mu_P = \{\mu_1,\ldots,\mu_\ell\}^T & & & & & \\
\mu_N = \{\mu_{\ell+1},\ldots,\mu_n\}^T & & & & &
\end{array}\right.
$$

where μ_j is the mean return of $\tilde{X}_j$, σ_{jj} its variance, and σ_{jk} its covariance with $\tilde{X}_k$; A is the variance-covariance matrix of future returns of all existing firms; μ_P and μ_N are the vectors of mean returns of the private and public sector firms respectively; B and C are the variance-covariance matrices of future returns of firms in the private and public sector respectively; and, finally, D is the matrix of covariances between the future returns of the private and public sector firms.

V.4 Private capital market. Investors have opportunity to borrow and lend in unlimited amounts at a given risk-free rate of interest denoted by (r - 1). The market is perfect.

2. Portfolio demand

The model derives equilibrium values p_j^*, $j \in P$, for the equity of the ℓ private firms at the beginning of the period. The demand of investor i, given an arbitrary value vector $p_P = \{p_1, \dots, p_\ell\}^T$, can be derived as follows:

Since he comes to the market with wealth Wi in the form of bonds and private share endowments, investor i's budget constraint is

(2) $$W_i = \bar{m}_i + \bar{z}_i^T p_P = m_i + z_i^T p_P \quad , \forall i \in H$$

where $\bar{z}_i^T = \{\bar{z}_{i1}, \dots, \bar{z}_{i\ell}\}$ and $z_i^T = \{z_{i1}, \dots, z_{i\ell}\}$ are vectors of initial and desired proportionate holdings of the ℓ private firms respectively. The initial and desired proportionate holdings of bonds are $\bar{m}_i$ and m_i respectively.

Investor i's objective is to maximize a function

(3) $$U_i = f_i(E_i, V_i) \quad , \forall i \in H$$

Risk aversion implies $f_{iE} > 0$, $f_{iV} < 0$.

Mean end of period wealth is given by

$$E_i = r m_i + z_i^T \mu_P + \alpha_i^T \mu_N \qquad \forall i \in H$$

which after substituting (2) is

(4) $$E_i = r\left[\bar{m}_i + \left(\bar{z}_i^T - z_i^T\right) p_P\right] + z_i^T \mu_P + \alpha_i^T \mu_N \qquad \forall i \in H$$

The variance of end of period wealth is

$$(5) \quad V_i = z_i^T B z_i + \alpha_i^T C \alpha_i + 2 z_i^T D \alpha_i \qquad \forall i \in H$$

Substituting (4) and (5) into (3) yields the objective function

$$(6) \quad U_i = f_i \{ r[\bar{m}_i + (\bar{z}_i^T - z_i^T) p_P] + z_i^T \mu_P + \alpha_i^T \mu_N \,;\, z_i^T B z_i + \alpha_i^T C \alpha_i + 2 z_i^T D \alpha_i \} \qquad \forall i \in H$$

whose FOC for maximum of utility with respect to the demand vector is

$$(7) \quad dU_i / dz_i = f_{iE} (dE_i / dz_i) + f_{iV} (dV_i / dz_i)$$
$$= f_{iE} (\mu_P - r p_P) + 2 f_{iV} (B z_i + D \alpha_i) = 0 \qquad \forall i \in H$$

The optimal demand vector is the solution of the set of equations represented by (7):

$$(8) \quad z_i = (f_{iE} / -2 f_{iV}) B^{-1} (\mu_P - r p_P) - B^{-1} D \alpha_i \qquad \forall i \in H$$

Basic remark 1.- Equation (8) violates the portfolio separation theorem of the basic CAPM[12] and portfolio holdings are unique to the investor i. Intuitively, investors' desired holdings of private sector firms' stocks are adjusted according to their holdings of public sector firms that they are constrained to hold, based on the covariance structure of the returns of the firms in the two sectors.

3. Market equilibrium

Market equilibrium can be defined as a vector of values p_P^* at which excess demands for all I private stocks are zero, where demands for the H investors are those which maximize their respective utilities.

Imposing the market clearing condition for the private sector firms

$$(9) \quad \sum_i z_i = \hat{1}$$

and a similar condition for the public sector

$$(10) \quad \sum_i \alpha_i = \hat{1}$$

where $\hat{1}$ is the unit vector, in equation (8) yields

$$(11) \quad p_P^* = (1/r) [\mu_P - \mathcal{R} (B \hat{1} + D \hat{1})]$$

where

$$(12) \quad \mathcal{R} \equiv 1 / \sum_i (f_{iE} / -2 f_{iV})$$

is the market risk aversion parameter.

The price for the jth private sector firm can therefore be written as

(13) $$p_j^* = (1/r)\left[\mu_j - R\left(\sigma_{jj} + \sum_{k\neq j}^{n} \sigma_{jk}\right)\right] \qquad j = 1,\dots,\ell$$

Equation (13) says that the value of the jth private firm is the discounted value at the risk free rate of interest of the certainty equivalent of the firm's return. This certainty equivalent is the expected value of the return less its variance plus the sum of the covariances with all other (private and public) returns in the economy multiplied by R, the market price of risk[13].

Basic remark 2.- The values of firms in the private sector of the mixed economy, as a given by equation (13), are identical to those in the CAPM pure private economy[14].

4. Equilibrium portfolios

The (sub)optimal[15] portfolio of private assets held by individual i, Z_i^*, can be obtained by substituting (11) into (8)

(14) $$Z_i^* = (\beta_{iE}/-2\beta_{iV})R\left(\hat{1} + B^{-1}D\,\hat{1}\right) - B^{-1}D\,\alpha_i \qquad \forall i \in H$$

This expression is stated in terms of market data and deserves the same comment as that given to equation (8).

5.3 A two-by-two mixed economy.

We consider now the specific case of a mixed economy in which there exist only two firms j= 1 (private), 2 (public) and two investors i=1, 2. This is the framework where the last contributions to the debate have emerged. Below are given on the left (right) Holmstrom's (Stapleton's and Subrahmanyam's) formulations.

Utility function of individual i over his final uncertain wealth is of the CARA type

(15) $$u_i(\tilde{Y}_i) = -\beta_i \exp(-\tilde{Y}_i/\beta_i) \quad \Big| \quad -\beta_i \exp(-\tilde{Y}_i\beta_i) \qquad i = 1,2$$

which can be written in mean-variance terms as

(16) $$U_i(E_i, V_i) = E_i - (1/2\beta_i)V_i \quad \Big| \quad E_i - (\beta_i/2)V_i \qquad i = 1,2$$

where (in scalar terms)

(17) $$E_i = r\left[\bar{m}_i + (\bar{z}_i - z_i)p_1\right] + z_i\mu_1 + \alpha_i\mu_2 \qquad i=1,2$$

(18) $$V_i = z_i^2\sigma_{11} + \alpha_i^2\sigma_{22} + 2z_i\alpha_i\sigma_{12} \qquad i=1,2$$

Individual i's risk tolerance is

(19) $$(\rho_{iE}/-2\rho_{iV}) = \beta_i \quad \vdots \quad (1/\beta_i) \qquad i=1,2$$

and the market risk aversion parameter is given therefore by

(20) $$R = 1/\textstyle\sum_i (\rho_{iE}/-2\rho_{iV}) = 1/\sum_i \beta_i \quad \vdots \quad 1/\sum_i (1/\beta_i) \qquad i=1,2$$

Noticing that B= σ_{11} , B^{-1}= ($1/\sigma_{11}$), C= σ_{22} , D= σ_{12} , $B^{-1}D$= ($1/\sigma_{11}$) σ_{12} , equations (8), (13) and (14) reduce to the following scalar expressions

(21) $$z_i = (1/\sigma_{11})\left[(\mu_1 - rp_1)\beta_i - \sigma_{12}\alpha_i\right] \quad \vdots \quad (1/\sigma_{11})\left[(\mu_1 - rp_1)(1/\beta_i) - \sigma_{12}\alpha_i\right] \qquad i=1,2$$

(22) $$p_1^* = (1/r)\left[\mu_1 - R(\sigma_{11} + \sigma_{12})\right]$$

(23) $$z_i^* = \beta_i R + (\beta_i R - \alpha_i)(\sigma_{12}/\sigma_{11}) \quad \vdots \quad (1/\beta_i)R + \left[(1/\beta_i)R - \alpha_i\right](\sigma_{12}/\sigma_{11}) \qquad i=1,2$$

To repeat, equation (21) gives the demand for private shares of individual i. Equation (22) yields the equilibrium value of private firm's stock. Finally, equation (23) gives investor i's purchase of private shares in terms of given data.

Remark.- As commented in the previous section the separation theorem does not hold in this economy. In a PPE agents would purchase shares $\hat{z}_i = \alpha_i = \beta_i R$, i=1,2 (taking the Holmstrom's case). This situation would not come out in the present model except by accident provided σ_{12}, σ_{11} are different from zero and finite. Investor i, however, can offset his biased allocation of public firm's shares when there is perfect correlation between the two firms. As an illustration, assume the $\tilde{X}_1 = b\tilde{X}_2$, b=constant. Then, applying (22) i's return will be

(24) $$z_i^*\tilde{X}_1 + \alpha_i\tilde{X}_2 = \beta_i R(1+b)\tilde{X}_2$$

which is the same return he would obtain if both firms were marketed. Notice that Sandmo's analysis corresponds to this special case.

5.4 Stapleton's and Subrahmanyam's optimal investment rules

These authors compare discount rates in a mixed economy whith those in an identical but purely private economy (PPE). Their contribution can be summarized as follows.

SS1. Financing of private investment is made through a reduction in the amount of riskless investment.

SS2. Firms j= 1,2 use different technologies characterized by stochastic constant returns to scale, so that we can write (1) in per-unit investment terms as

$$(25)\quad \begin{cases} \tilde{x}_j = \tilde{X}_j / I_j & j=1,2 \\ \mu'_j = \mu_j / I_j & " \\ \sigma'_{jj} = \sigma_{jj} / I_j^2 & " \\ \sigma'_{jk} = \sigma_{jk} / I_j\, I_k = \sigma'_j\, \sigma'_k\, \rho_{jk} & j \neq k=1,2 \end{cases}$$

Where ρ_{jk} is the correlation coefficient between the two technologies

SS3. Pareto optimal levels of investment in a PPE are given by

$$(26)\quad \hat{I}_j = (1/R\,\sigma'_{jj})[\mu'_j - r - RI_k\,\sigma'_{jk}] \qquad j\neq k=1,2$$

and achieved by wealth[16]-maximizing firms acting as price takers. These levels of investment obtain when firms equalize at the margin their expected return per unit of investment, μ'_j, with their cost of capital, $\hat{\rho}_j$, which are given by

$$(27)\quad \hat{\rho}_j = r + R\,(I_j\,\sigma'_{jj} + I_k\,\sigma'_{jk}) \qquad j\neq k=1,2$$

SS4. Firm 2 is nationalized and, by the analysis made in the previous two sections, the value of firm 1 does not change. Taking advantage of (22) and (25) we can write it as

$$(28)\quad \Gamma_1^* = (1/r)[\, I_1\mu'_1 - R\,(I^2\,\sigma'_{11} - I_1 I_2\,\sigma'_{12})]$$

Proposition 1 (Optimal Investment in the Private Sector) Wealth-maximizing investment decisions of the private sector firm are the same as they would be in a PPE.

Proof.- Taking advantage of expressions (16) through (21) and (28), it can be shown that

$$(29) \qquad dU_i/dI_1 = (R/\beta_i)[\mu_1' - r - R(I_1 \sigma_{11}' + I_2 \sigma_{12}') \qquad i=1,2$$

which evaluated at $I_1 = \hat{I}_1$ in (26) is $(dU_i/dI_1)\big|_{I_1=\hat{I}_1} = 0$ $\quad$ i=1,2

Remark.- This result depends on the level of investment in the public firm remaining unchanged

Proposition 2 (Optimal Investment in the Public Sector). The cost of capital to the public firm, the social rate of discount, is higher than it would be if the firm were private.

Proof.- Step 1 (Pareto criterion failure). Taking again advantage of expression (16) through (21) and (28), it can be shown that

$$(30) \qquad (dU_i/dI_2)\big|_{I_1=\hat{I}_1} = \alpha_i[\mu_2' - r - \beta_i \alpha_i I_2 \sigma_{22}'] + [\sigma_{12}'/\sigma_{11}'] \cdot [R/\beta_i - \alpha_i][R(I_1 \sigma_{11}' + I_2 \sigma_{12}') - \beta_i \alpha_i I_2 \sigma_{12}'] + \sigma_{12}'[(-R^2/\beta_i) I_1 + (r/\sigma_{11}')(R/\beta_i - \alpha_i)]$$

Noticing that $\alpha_2 = 1 - \alpha_1$,

if $\alpha_1 \ (\gtrless) \ R/\beta_1 \Rightarrow (dU_1/dI_2)\big|_{I_1=\hat{I}_1} \ (\lessgtr) \ 0$ and $(dU_2/dI_2)\big|_{I_1=\hat{I}_1} (\gtrless) 0$,

and viceversa. In words, if the share of public firm allocated to individual i (i=1,2) differs from his optimal holding in the PPE, i.e, $\alpha_i \neq R/\beta_i$, then one individual will desire increase investment and the other a reduction in the level of investment after nationalization has taken place.

Step 2 (Kaldor-Hicks criterion)

$$(31) \qquad \sum_i (dU_i/dI_2) = (\alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 - R) I_1 \cdot [-\sigma_{11}' + (\sigma_{12}'/\sigma_{22}')^2] \leq 0$$

with equality only if $\alpha_i = R/\beta_i$ and or $q_{12} = \pm 1$[17]. Apart from these cases the derivative in (31) is strictly negative indicating that a reduction in the level of investment I_2 would increase welfare. That is, while the level of investment in the private sector $I_1 = \hat{I}_1$ remains Pareto optimal in the mixed economy, the level of investment in the public sector should be reduced from its level $I_2 = \hat{I}_2$ in the PPE. This implies a higher cost of capital for the public firm, the expression of which can be derived by setting (31) equal to zero and solving for μ'_2 to get:

$$(32)\quad \mu'_2 = \rho_2 = r + R\,(I_1 \sigma'_{12} \pm I_2 \sigma'_{22}) + (\alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 - R) \cdot \left[\sigma'_{22} - (\sigma'_{12}/\sigma'_{11})^2\right]$$

$$= \hat{\rho}_2 + (\alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 - R)\left[\sigma'_{22} - (\sigma'_{12}/\sigma'_{11})^2\right] \geq \hat{\rho}_2$$

5.5 Holmstrom's optimal investment rules

Holmstrom's starting point is the situation of financial equilibrium described in section 5.3. He then introduces for both firms the same opportunity to invest in a new project and examines whether the public firm should require a higher or lower expected return than the private one. The following assumptions are made:

H.1 Financing of the new project is made (when privately undertaken) through a reduction in the amount of riskless investment .

H.2 Return on investment in the new opportunity is of stochastically constant returns to scale type, i.e, and end of period return $\tilde{Z}$ per unit of investment. It is postulated that $\tilde{Z}$ is normally distributed with mean μ_z, variance σ_{zz} and covariances $\sigma_{jz} = \mathrm{cov}\,(\tilde{X}_j, \tilde{Z})$, j=1,2.

H.3 Investments in both firms are judged according to the Pareto criterion[19], which recomends investment if

$$\left(\textstyle\sum_i dU_i / dI\right)\Big|_{I=0} > 0$$

Optimal investment in the private sector

Assume that the new project is undertaken by the private firm[20]. When firm 1 invests I in the project, if we distinguish the prior (preinvestment) variables by * we obtain the new variables

$$(33)\quad \begin{cases} \tilde{X}_1 = \tilde{X}_1^* + I\tilde{Z} \\ \mu_1 = \mu_1^* + I\,\mu_z \\ \sigma_{11} = \sigma_{11}^* + 2I\,\sigma_{1z} + I^2\,\sigma_{zz} \\ \sigma_{12} = \sigma_{12}^* + I\,\sigma_{2z} \\ Z_i = (1/\sigma_{11})\left[(\mu_1 - r\,p_1)\,\beta_i - \sigma_{12}\,\alpha_i\right], \; i=1,2 \\ p_1 = (1/r)\left[\mu_1 - R\,(\sigma_{11} + \sigma_{12})\right. \end{cases}$$

The following facts will be used

$$(34)\quad \begin{cases} (d\mu_1 / dI)\big|_{I=0} = \mu_z \\ (d\,\sigma_{11} / dI)\big|_{I=0} = 2\,\sigma_{1z} \\ (d\,\sigma_{12} / dI)\big|_{I=0} = \sigma_{2z} \end{cases}$$

In order to get the Pareto criterion we must compute $(dU_i/dI)\big|_{I=0}$. Taking advantage of (16) we have

$$(35)\quad (dU_i/dI)\big|_{I=0} = (dE_i/dI)\big|_{I=0} - (1/2\beta_i)\,(dV_i/dI)\big|_{I=0}$$

Now, since we start from a position of equilibrium, taking advantage of (17), (18), (33) and (34), replacing $^-$ by * in (17), and noticing that variables at zero investment levels coincide with preinvestment equilibrium variables, we obtain

$$(36)\quad (dE_i/dI)\big|_{I=0} = \left(d\left[\,r\left\{m_i^* + (Z_i^* - Z_i)\,p_1\right\} + Z_i\,\mu_1 + \alpha_i\,\mu_2^*\right]/dI\right)\big|_{I=0}$$
$$= (\mu_1^* - r\,p_1^*)\,(dZ_i/dI)\big|_{I=0} + Z_i^*\,\mu_z$$

$$(37)\quad (dV_i/dI)\big|_{I=0} = \left(d\left[\,Z_i^2\,\sigma_{11} + \alpha_i^2\,\sigma_{22}^* + 2Z_i\,\alpha_i\,\sigma_{12}\right]/dI\right)\big|_{I=0}$$
$$= 2\left[\,Z_i^{*2}\,\sigma_{1z} + Z_i^*\,\alpha_i\,\sigma_{2z} + (Z_i^*\,\sigma_{11}^* + \alpha_i\,\sigma_{12}^*)\cdot (dZ_i/dI)\big|_{I=0}\right]$$

Substituting (36) and (37) into (35) one gets[21]

$$(38)\quad (dU_i/dI)\big|_{I=0} = Z_i^*\,\mu_z - (1/\beta_i)\,(Z_i^{*2}\,\sigma_{1z} + Z_i^*\,\alpha_i\,\sigma_{2z})$$
$$+\left[\,\mu_1^* - r\,p_1^* - (1/\beta_i)\,(Z_i^*\,\sigma_{11}^* + \alpha_i\,\sigma_{12}^*)\right]\cdot (dZ_i/dI)\big|_{I=0}$$

$$= Z_i^* \mu_z - (1/\beta_i)\,(Z_i^{*2}\sigma_{1z} + Z_i^* \alpha_i \sigma_{2z})$$

since applying (21) the expression in brackets cancels out.

Adding (38) over the two investors and making use of conditions $\sum_i Z_i^* = 1$, and $\sum_i \alpha_i = 1$ yields the <u>Pareto criterion for private investment</u> in the new project

$$(39) \qquad \sum_i (d\,U_i/dI)\Big|_{I=0} = \mu_z - \rho_1 > 0$$

where

$$(40) \qquad \rho_1 \equiv R_1 \sigma_{1z} + R_2 \sigma_{2z}$$

$$(41) \qquad R_1 \equiv Z_1^{*2}/\beta_1 + Z_2^{*2}/\beta_2$$

$$(42) \qquad R_2 = Z_1^* \alpha_1/\beta_1 + Z_2^* \alpha_2/\beta_2$$

are, respectively, the cost of capital for the private firm and the prices that it associates with risk dimensions σ_{1z} and σ_{2z}.

As a standard of comparison, the <u>value-maximizing criterion</u>[22] would read

$$(43) \qquad \mu_z - \rho > 0$$

where

$$(44) \qquad \rho \equiv R\,(\sigma_{1z} + \sigma_{2z})$$

$$(45) \qquad R \equiv 1/(\beta_1 + \beta_2)$$

are, respectively, "the" cost of capital and "the" price of risk.

<u>Optimal Investment in the Public Sector</u>

Assume now that the project is undertaken by the public firm. It can be shown[23] that

$$(46) \qquad (d\,U_i/dI)\Big|_{I=0} = \alpha_i \mu_z - (1/\beta_i)\left[\alpha_i^2 \sigma_{2z} + Z_i^* \alpha_i \sigma_{1z}\right] \quad i=1,2$$

The <u>Pareto criterion for public investment</u>, which obtains adding (46) over the two investors and taking again advantage of conditions $\sum_i \alpha_i = 1$, $\sum_i Z_i^* = 1$, reads

$$(47) \qquad \sum_{i=1}^{2} (d\,U_i/dI)\Big|_{I=0} = \mu_z - \rho_2 > 0$$

where

(48) $\rho_2 = R_2\, \sigma_{1z} + R_3\, \sigma_{2z}$

R_2 as defined in (42)

(49) $R_3 \equiv \alpha_1^2/\beta_1 + \alpha_2^2/\beta_2$

are, respectively, the cost of capital for the public firm and the prices that it associates with risk dimensions σ_{1z} and σ_{2z}. Notice that public firm associates R_2 with σ_{1z} while private firm associates it with σ_{2z}.

The following Lemma and Propositions summarize the most important features of the model[24, 25]. In short, there is neither a single measure for project risk, nor a market price by which the cost of risk can be evaluated, and hence firms face a portfolio choice problem. Moreover, contrary to Stapleton's and Subrahmanyam's result "no unambiguous comparisons of the cost of capital between marketed firms seems possible when the two sectors are imperfectly correlated".

Lemma.- The following relationships between risk prices can be recorded

L.1: $R_1 > R$

L.2: $\sigma_{12} \ (\gtrless)\ 0 \Rightarrow R_2 \ (\lessgtr)\ R$

L.3: $R_3 > R$

L.4: $\sigma_{12} > 0$, $\sigma_{12} \ (\gtrless)\ \sigma_{11} \Rightarrow R_3 \ (\lessgtr)\ R_1$

L.5: $\alpha_i = Z_i^* = \beta_i R \ \ (i=1,2) \Rightarrow R = R_1 = R_2 = R_3$.

Remark.- In L.1 through L.4 the assumption is made that shares distribution is not optimal, i.e., the LHS of L.5 is not verified. Notice also that only when (accidentally) $\alpha_i = \beta_i R$, we have a unique price for risk.

Proof: See Appendix 1.

Proposition 1.- Taking advantage of the above Lemma, statements a and b below show that the Pareto criterion may be either stricter or more lenient than the value-maximization criterion for the private firm:

a. $\sigma_{12} > 0,\ \sigma_{1z} > 0,\ \sigma_{2z} = 0 \Rightarrow \rho_1 > \rho$

b. $\sigma_{12} > 0,\ \sigma_{1z} = 0,\ \sigma_{2z} > 0 \Rightarrow \rho_1 < \rho$

Proof: See Appendix 2.

In a mixed economy, however, the arbitrary allocation of public firm's "shares" through the tax system makes inappropriate the value-maximization criterion for investment [26]. The relevant comparison is, on the contrary, between ρ_1 and ρ_2. Some summary statements are given by the following

Proposition 2.- The cost of capital can in general be either higher or lower in the private (marketed) sector than in the public (non-marketed) sector. As an illustration, assume $\sigma_{12} > 0$, $\alpha_1 \neq \beta_1 R$. Then

a. $\sigma_{1z} \leq 0,\ \sigma_{2z} > 0 \Rightarrow \rho_1 < \rho_2$

b. $\sigma_{1z} > 0,\ \sigma_{2z} \leq 0 \Rightarrow \rho_1 > \rho_2$

c. for any σ_{1z}, there exists a $\bar{\sigma}_{2z}$ such that $\sigma_{2z} \ (\gtrless)\ \bar{\sigma}_{2z} \Rightarrow \rho_1 (\lessgtr) \rho_2$.

d. for any σ_{2z}, there exists a $\bar{\sigma}_{1z}$ such that $\sigma_{1z} \ (\gtrless)\ \bar{\sigma}_{1z} \Rightarrow \rho_1 (\gtrless) \rho_2$.

e. $\sigma_{1z} > \sigma_{2z} > 0,\ \sigma_{11} > \sigma_{12} \not\Rightarrow \rho_1 < \rho_2$

e′. $\sigma_{1z} = \sigma_{2z} > 0,\ \sigma_{11} > \sigma_{12} \Rightarrow \rho_1 < \rho_2$

e′′. $\sigma_{2z} > \sigma_{1z} > 0,\ \sigma_{11} > \sigma_{12} \Rightarrow \rho_1 < \rho_2$

f. (Independent of the heading assumption) $\sigma_{1z} = \sigma_{2z} = 0 \Rightarrow \rho_1 = \rho_2 = 0$

Remark.- As shown in Appendix 3, statement e is not valid without further qualifications. I have included statements e′ and e′′ which serve, I hope, the same purpose.

Proof: See Appendix 3

Statements a and b show that diversification pays for both firms (b can be taken as nn illustration of the Sammuelson-Vickrey view). Consideration of market structure and the type of risk present in different sectors alters Hirshleifer's recommendation that government should imitate private investment. Notice also that even if (public) firm 2 contemplates an investment in the same risk class as firm 1 (so that $\sigma_{1z} \propto \sigma_{11}$, $\sigma_{22} \propto \sigma_{12}$) it cannot read the price of risk from (22)[27] - since R_2, $R_3 \neq R$ -, nor will use the same criterion as firm 1 would use in considering the same project[28].

Statements c and d show monotonocity properties. The higher, say, σ_{1z}, the more likely the cost of capital for firm 1 will - ceteris paribus - be higher than for firm 2.

Statements e′ and e′′ show that the size of firms also play a role. If the private sector is large, $\sigma_{12} < \sigma_{11}$, $\sigma_{2z} \geq \sigma_{1z}$, and this makes the cost of capital in that sector relatively lower. Reversing the inequalities one gets again Samuelson - Vickrey's results.

Finally statement f is included to represent the Arrow-Lind case, i.e., a new project uncorrelated with both sectors. The absence of risk[29] makes no distinction between ρ_1 and ρ_2, which are in fact zero. Therefore, we have a situation of imperfect risk sharing in the private sector and an uncorrelated project but no argument to recomend $\rho_1 > \rho_2 = 0$. This is a conclusion that Holmstrom does not want to draw, but it is clear.

FOOTNOTES

1. In the state-contingent commodity model contracts for the transfer of a commodity specify not only its physical properties and the date of delivery, but also an event on which the transfer is conditional. Preferences, production, and endowments are then specified in terms of such contingent commodities, and a market is created for each such commodity.

2. In the perfect insurance markets, prices or "risk premia, would be associated with each competitive equilibrium allocation indicating the terms on wich individuals would be willing to trade certain income for risky outcomes.

3. In the Arrow-Debreu economy, if the returns of a public sector investment project are uncorrelated with the value of national income (measured as excluding the returns to the project), then the relevant public sector discount rate does not contain a risk premium.

4. Moral hazards, transaction costs... are often quoted as reasons why markets for many types of insurance do no exist.

5. Risk-pooling occurs when an investor has a portfolio of <u>many small</u> projects with (relatively) <u>independent</u> probability distributions. By the principles of portfolio selection, the probability distribution of the portfolio is much less dispersed than the sum of the distributions of individual projects, and the cost of risk bearing will be low.

6. See Arrow-Lind (1972)

7. This is also valid for Sadmon's work, although he does not use the CAPM framework.

8. The proportionality assumption represents a particular specification of the firm as a price taker. It says that with a production function such that output will change by the same percentage in every state of the world, the firm will base its investment decision on the belief that the value of the firm will change by the same percentage. See Mossin, Chap. 13.

9. This Section is an adaptation of Stapleton and Subrahmanyam (1980), Chaps. I and VII.

10. If the final returns of the public firms are high (low), the taxes levied on individuals are relatively low (high).

11. Constant Absolute Risk Aversion.

12. In a pure private economy (PPE) equation (8) would read

$$Z_i = (f_{iE} / -2 f_{iY})\, A^{-1} (\mu - r \rho)$$

where A is defined in (1), μ is the vector of mean returns and ρ is the vector of values. In such an economy each investor chooses a multiple $(f_{iE} / -2f_{iY})$ of a standard vector of portfolio proportions (the market portfolio)

13. We shall see later that, apart from the expression giving equilibrium values, no such a market price of risk may exist when investment is considered.

14. In the PPE equations (11) and (13) can be obtained by imposing the market clearing condition $\sum_i Z_i = \hat{1}$, in the equation appearing in fn 12.

15. Suboptimal relative to the PPE case in which (14) would read $Z_i^* = (- f_{iE} / 2f_{iY})\, R\, \hat{1} \quad \forall\, i \in H$

16. These authors use wealth-maximizing and value-maximizing expressions as equivalent.

17. We have to show that $(\alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 - R) > 0$ provided $\alpha_i \neq R/\beta_i$ since $I_1 > 0$ and $[\quad] \leq 0$. Choosing $\alpha_1 = R/\beta_1 + \varepsilon \Rightarrow \alpha_2 = R/\beta_2 - \varepsilon$. Then, $\alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 = [R/\beta_1 + \varepsilon]^2 \beta_1 + [R/\beta_2 - \varepsilon]^2 \beta_2 = R + \varepsilon^2 (\beta_1 + \beta_2) > R$ provided $\varepsilon \neq 0$.

19. This title is misleading. In the literature on Pareto optimality of investment in private capital markets (see Mossin, Chap. 15) the Pareto criterion requires $(dU_i / dI)\big|_{I=0} > 0 \ \forall_i$. This last condition is not verified in the present model so that what Holmstrom is actually using is the Kaldor-Hicks criterion. We preserve, however, Holmstrom's term.

20. In the formulation that follows monopolistic behaviour is postulated on the part of both firms with respect to the new project. I make explicit here a hidden and contradictory assumption made by Holmstrom. See also fn 22 below.

21. Notice that expressions $(d p_1 / dI)\big|_{I=0}$, $(dZ_i / dI)\big|_{I=0}$ play no role. Note also that neither r nor σ_{11} appear.

22. This statement needs the qualifier "in a competitive PPE" for it is clear that from (33)

$$(d p_1/dI)\big|_{I=0} = (1/r)[\mu_z - R(2\sigma_{1z} + \sigma_{2z})] > 1 \Rightarrow$$
$$[\mu_z - R(2\sigma_{1z} + \sigma_{2z})] > r > 0 \qquad (14.1)$$

while in a competitive PPE, the value maximizing criterion (which is Pareto optimal) would read

$$(d p_1/dI)\big|_{I=0} = (1/r)[\mu_z - R(\sigma_{1z} + \sigma_{2z})] > 1 \Rightarrow$$
$$[\mu_z - R(\sigma_{1z} + \sigma_{2z})] > r > 0 \qquad (14.2)$$

The word competitive is crucial since monopolistic undertaking of the new project even in a PPE prevents reaching (first-best relative to Holmstrom concept of) Pareto optimality. The difference between (14.1) and (14.2) reflects the kind of monopolistic behaviour we refered to in fn 20. For further details see Mossin, chap 15. In spite of this we preserve, as before, Holmstrom's term.

23. See Appendix 4 for the derivation.

24/5 The formulation presented above differs slightly from Holmstrom's. This has been due to, say, the lack of clarity of the following statements "... (47)... gives the cost of capital for the nonmarketed from, and (39)... gives the cost of capital for the marketed firm". It is clear that at most these expressions give implicitely the costs of capital. The RHS's of (47) and (39) are only benefit differentials at the margin. Welfare is maximized by just setting them equal to zero to yield the optimal (public and private) levels of investment. The corollary is that the new investment should be undertaken by the firm (public or private) which maximizes $\sum_i U_i$.

26. Since private shares are not held so as to minimize risk.

27. As occurs even with the private firm.

28. Since risk measures (σ_{1z}, σ_{2z}) are evaluated at prices (R_1, R_2) by the private firm, while (R_2, R_3) are the prices which the public firm would use.

29. Recall that the variance of the project is not contemplated in ρ_j formulas.

Appendix 1. Proof of Lemma

The following relationships will be used:

$$R_1 = (Z_1^{*2} / \beta_1) + (Z_2^{*2} / \beta_2)$$

$$R_2 = (Z_1^* \alpha_1 / \beta_1) + (Z_2^* \alpha_2 / \beta_2)$$

$$R_3 = (\alpha_1^2 / \beta_1) + (\alpha_2^2 / \beta_2)$$

$$R = 1 / \bar{\beta} \ ; \ \bar{\beta} = \beta_1 + \beta_2$$

where

$Z_i^* = \beta_i R + (\beta_i R - \alpha_i)(\sigma_{12}/\sigma_{11})$, $i=1,2$ (adjusted asset holdings in terms of market parameters)

α_i, $i=1,2$ exogenously given

Remark.- $\alpha_i = \beta_i R \Rightarrow Z_i^* = \beta_i R$; $Z_i^* = \beta_i R \Rightarrow \alpha_i = \beta_i R$ provided $\sigma_{12} \neq 0$ (a condition that we shall safely assume).

L.1 $Z_i^* \neq \beta_i R \Rightarrow R_1 > R$ (deleting *)

$R_1 = (Z_1^2 / \beta_1) + (Z_2^2 / \beta_2)$ and using $Z_2 = (1 - Z_1)$

$$R_1(Z_1) = (Z_1^2 / \beta_1) + [(1 - Z_1)^2 / \beta_2]$$

$$R_1'(Z_1) = 2[(Z_1 / \beta_1) + (Z_1 / \beta_2) - (1/\beta_2)] = 0 \Rightarrow \hat{Z}_1 = \beta_1 R$$

$$R_1''(Z_1) = 2[(1/\beta_1) + (1/\beta_2)] > 0$$

R_1' and R_1'' assure that $R_1(Z_1)$ reaches its minimum at $\hat{Z}_1 = \beta_1 R$.

But $R_1(\hat{Z}_1) = R$ so that $Z_1^* \neq \hat{Z}_1 \Rightarrow R_1 > R$ (Q.E.D)

On the contrary, $Z_1^* = \hat{Z}_1 \Rightarrow R_1 = R$, which proves the first part of L.5 (Q.E.D)

L.2 $\alpha_i \neq \beta_i R$, $\sigma_{12} (\gtrless) 0 \Rightarrow R_2 (\lessgtr) R$

$R_2 = (Z_1 \alpha_1 / \beta_1) + (Z_2 \alpha_2 / \beta_2)$, and using $\alpha_2 = 1 - \alpha_1$;

$Z_2 = 1 - Z_1$; $Z_1 \equiv Z_1(\alpha_1) = Z_1^*$

$$R_2(\alpha_1) = [(Z_1(\alpha_1)\alpha_1 / \beta_1) + ((1 - Z_1(\alpha_1))(1 - \alpha_1)/\beta_2)]$$

$R_2'(\alpha_1) = (2/\beta_1\beta_2)(\sigma_{12}/\sigma_{11})(\beta_1 - \alpha_1\bar{\beta}) = 0 \Rightarrow \hat{\alpha}_1 = \beta_1 R$

$R_2''(\alpha_1) = -(2\bar{\beta}/\beta_1\beta_2)(\sigma_{12}/\sigma_{11}) \quad (\lessgtr) \text{ if } \sigma_{12} (\gtrless) 0$

Therefore $\sigma_{12} (\gtrless) 0$ assures that $R_2(\alpha_1)$ reaches its (maximum / minimum) at $\hat{\alpha}_1 = \beta_1 R$, where

$R_2(\hat{\alpha}_1) = R$. Thus, $\alpha_1 \neq \hat{\alpha}_1 \Rightarrow R_2 (\lessgtr) R$ (Q.E.D.)

On the contrary $\alpha_1 = \hat{\alpha}_1 \Rightarrow R_2 = R$, which proves the second part of L.5 (Q.E.D.)

<u>L.3</u> $\alpha_i \neq \beta_i R \Rightarrow R_3 > R$

$R_3 = (\alpha_1^2/\beta_1) + (\alpha_2^2/\beta_2)$ and using $\alpha_2 = 1 - \alpha_1$

$R_3(\alpha_1) = (\alpha_1^2/\beta_1) + [(1-\alpha_1)^2/\beta_2]$

$R_3'(\alpha_1) = 2[(\alpha_1/\beta_1) + (\alpha_1/\beta_2) - (1/\beta_2)] = 0 \Rightarrow \hat{\alpha}_1 = \beta_1 R$

$R_3''(\alpha_1) = 2[(1/\beta_1) + (1/\beta_2)] > 0$

R_3' and R_3'' assure that $R_3(\alpha_1)$ reaches its minimum at $\hat{\alpha}_1 = \beta_1 R$,

Where $R_3(\hat{\alpha}_1) = R$.

Thus, $\alpha_1 \neq \hat{\alpha}_1 \Rightarrow R_3 > R$. (Q.E.D.)

On the contrary, $\alpha_1 = \hat{\alpha}_1 \Rightarrow R_3 = R$, which prove_s the third part of L.5 (Q.E.D.)

<u>L.4</u> $\alpha_i \neq \beta_i R$, $\sigma_{12} > 0$, $\sigma_{12} (\gtrless) \sigma_{11} \Rightarrow R_3 (\lessgtr) R_1$

Let define $\Delta \equiv R_3 - R_1 = (\alpha_1^2/\beta_1) + (\alpha_2^2/\beta_2) - (z_1^2/\beta_1) - (z_2^2/\beta_2)$

deleting *, using $\alpha_2 = 1 - \alpha_1$, $z_2 = (1 - z_1)$, $z_1 \equiv z_1(\alpha_1)$

$$\Delta(\alpha_1) = (\alpha_1^2/\beta_1) + ((1-\alpha_1)^2/\beta_2) - ((1 - z_1(\alpha_1))^2/\beta_2) - (z_1^2(\alpha_1)/\beta_1)$$

$$\Delta'(\alpha_1) = 2\left\{\frac{\alpha_1}{\beta_1} - \frac{(1-\alpha_1)}{\beta_2} + \left[\frac{(1-z_1)}{\beta_2} - \frac{z_1}{\beta_1}\right]\frac{\partial z_1}{\partial\alpha_1}\right\},$$

and noticing that $\dfrac{\partial z_1}{\partial \alpha_1} = -\dfrac{\sigma_{12}}{\sigma_{11}}$

$$= (2 / \beta_1 \beta_2 R)\left[1 - (\sigma_{12}/\sigma_{11})^2\right](\alpha_1 - R\beta_1)$$

$$= 0 \Rightarrow \hat{\alpha}_1 = R\beta_1 \quad \text{provided } (\sigma_{12}/\sigma_{11}) \neq 1 .$$

$$\Delta''(\alpha_1) = (2/\beta_1\beta_2 R)\left[1 - (\sigma_{12}/\sigma_{11})^2\right] (\lessgtr) 0$$

$$\text{if} \quad \sigma_{12} (\gtrless) \sigma_{11} \quad \text{and } \sigma_{12} > 0 .$$

Therefore $\sigma_{12} > 0$, $\sigma_{12} (\gtrless) \sigma_{11}$ assures that $\Delta(\alpha_1)$ reaches its (maximum / minimum) at $\hat{\alpha}_1 = R\beta_1$, where

$\Delta(\hat{\alpha}_1) = 0 \Rightarrow R_3 = R_1$. Thus, $\alpha_1 \neq \hat{\alpha}_1$, $\sigma_{12} > 0$, $\sigma_{12} (\gtrless) \sigma_{11}$

$$\Rightarrow \Delta(\alpha_1) (\lessgtr) 0 \Rightarrow R_3 (\lessgtr) R_1 \qquad \text{(Q.E.D.)}$$

On the contrary, $\alpha_1 = \hat{\alpha}_1 \Rightarrow R_3 = R_1$

Appendix 2 Proof of Proposition 1

Define $\rho_1 - \rho \equiv \Delta = (R_1 - R)\,\sigma_{1z} + (R_2 - R)\,\sigma_{2z}$

(From L.1 $(R_1 - R) > 0$, and from L.2,

$\sigma_{12} > 0 \Rightarrow (R_2 - R) < 0$)

$= (+)\,\sigma_{1z} + (-)\,\sigma_{2z}$

(P - a) $\sigma_{12} > 0,\ \sigma_{1z} > 0,\ \sigma_{2z} = 0 \Rightarrow \rho_1 > \rho$

$\Delta = (+)\,(+) + (-)\,(0) = (+) \Rightarrow \rho_1 > \rho$

(P - b) $\sigma_{12} > 0,\ \sigma_{1z} = 0,\ \sigma_{2z} > 0 \Rightarrow \rho_1 < \rho$

$\Delta = (+)\,(0) + (-)\,(+) = (-) \Rightarrow \rho_1 < \rho$

Appendix 3. Proof of Proposition 2

The assumption $\sigma_{12} > 0$, $\alpha_1 \neq \beta_1 R$ permits to write the lemma as follows:

L.1: $R_1 > R$

L.2: $R > R_2$

L.3: $R_3 > R$

L'.4 (with $\sigma_{12} < \sigma_{11}$): $R_3 > R_1$

which yields the following inequalities

(A 3. 1) (without L'.4) $R_1 > R > R_2 \Rightarrow (R_1 - R_2) > 0$

(A 3. 2) (without L'.4) $R_3 > R > R_2 \Rightarrow (R_2 - R_3) < 0$

(A 3. 3) (with L'.4) $R_3 > R_1 > R > R_2 \Rightarrow (R_1 - R_2) > 0,$

$(R_2 - R_3) < 0, \ (R_1 - R_3) < 0$

Define now

(A 3. 4) $\Delta \equiv \rho_1 - \rho_2$

$$= (R_1 - R_2)\ \sigma_{1z} + (R_2 - R_3)\ \sigma_{2z} = \underset{A3.1}{(+)}\ \sigma_{1z} + \underset{A3.2}{(-)}\sigma_{2z}$$

We prove statements a through d using A3.1 and A3.2

(P - a) $\sigma_{1z} \lesseqgtr 0, \quad \sigma_{2z} > 0 \Rightarrow \rho_1 < \rho_2$

$\Delta \equiv (+)\ (\bar{\circ}) + (-)\ (+) = (-) \Rightarrow \rho_1 < \rho_2$

(P - b) $\sigma_{1z} > 0, \quad \sigma_{2z} \lesseqgtr 0 \Rightarrow \rho_1 > \rho_2$

$\Delta = (+)\ (+) + (-)\ (\bar{\circ}) = (+) \Rightarrow \rho_1 > \rho_2$

(P - c) $\forall\ \sigma_{1z}\ \exists\ \bar{\sigma}_{2z} : \ \sigma_{2z}\ (\gtrless)\ \bar{\sigma}_{2z} \Rightarrow \rho_1\ (\lessgtr)\ \rho_2$

$\Delta = 0 \Rightarrow \sigma_{2z} = \bar{\sigma}_{2z} = -\left[(R_1 - R_2)\ /\ (R_2 - R_3)\right]\sigma_{1z}$

$= (+)\ \sigma_{1z}$

Take $\sigma_{2z} = \bar{\sigma}_{2z} + \varepsilon$

$$\Delta = (R_2 - R_3)\,\varepsilon \qquad = (-)\,\varepsilon$$

$$\varepsilon\ (\gtrless)\ 0 \equiv \sigma_{2z}\ (\gtrless)\ \bar{\sigma}_{2z} \Rightarrow \Delta\ (\lesseqgtr)\ 0 \Rightarrow \rho_1\ (\lesseqgtr)\ \rho_2$$

<u>(P - d)</u> $\forall\, \sigma_{2z}\ \exists\ \bar{\sigma}_{1z} : \ \sigma_{1z}\ (\gtrless)\ \bar{\sigma}_{1z} \Rightarrow \rho_1\ (\gtrless)\ \rho_2$

$$\Delta = 0 \Rightarrow \sigma_{1z} = \bar{\sigma}_{1z} = -\left[(R_2 - R_3)\ /\ (R_1 - R_2)\right] \sigma_{2z}$$

$$= (+)\ \sigma_{2z}$$

Take $\sigma_{1z} = \bar{\sigma}_{1z} + \varepsilon$

$$\Delta = (R_1 - R_2)\varepsilon \qquad = (+)\ \varepsilon$$

$$\varepsilon(\gtrless)\ 0 \equiv \sigma_{1z}\ (\gtrless)\ \bar{\sigma}_{1z} \Rightarrow \Delta\ (\gtrless)\ 0 \Rightarrow \rho_1\ (\gtrless)\ \rho_2$$

<u>(P - e)</u> $\sigma_{1z} > \sigma_{2z} > 0\ ,\ \sigma_{11} > \sigma_{12} \not\Rightarrow \rho_1 < \rho_2$

$$\Delta = (R_1 - R_2)\ \sigma_{1z} + (R_2 - R_3)\ \sigma_{2z}$$

Take $\sigma_{1z} = \sigma_{2z} + \varepsilon\ ,\ \varepsilon > 0$

$$\Delta = (R_1 - R_3)\ \sigma_{2z} + (R_1 - R_2)\,\varepsilon \quad \text{and using (A3.3)}$$

$$= (-)\ (+) + (+)\ (+) \ = \text{undefined}$$

<u>(P - e')</u> $\sigma_{1z} = \sigma_{2z} > 0\ ,\ \sigma_{11} > \sigma_{12} \Rightarrow \rho_1 < \rho_2$

$$\Delta = (R_1 - R_2)\ \sigma_{1z} + (R_2 - R_3)\ \sigma_{1z}$$

$$= (R_1 - R_3)\ \sigma_{1z} \quad \text{and using (A3.3)}$$

$$= (-)\ (+) = (-) \Rightarrow \rho_1 < \rho_2$$

<u>(P - e'')</u> $\sigma_{2z} > \sigma_{1z} > 0\ ,\ \sigma_{11} > \sigma_{12} \Rightarrow \rho_1 < \rho_2$

$$\Delta = (R_1 - R_2)\ \sigma_{1z} + (R_2 - R_3)\ \sigma_{2z}$$

Take $\sigma_{2z} = \sigma_{1z} + \varepsilon\ ,\ \varepsilon > 0$

$$\Delta = (R_1 - R_3)\ \sigma_{1z} + (R_2 - R_3)\,\varepsilon \quad \text{and using (A3.3)}$$

$$= (-)\ (+) + (-)\ (+) = (-) \Rightarrow \rho_1 < \rho_2$$

<u>(P - f)</u> (Independent of the heading assumption)

$$\sigma_{1z} = \sigma_{2z} = 0 \Rightarrow \rho_1 = \rho_2 = 0$$

$$\Delta = 0 \Rightarrow \rho_1 = \rho_2$$

Further, since risk measures σ_{1z} and σ_{2z} are zero, $\rho_1 = \rho_2 = 0$. For example,

$$\rho_1 = R_1 \sigma_{1z} + R_2 \sigma_{2z} = (R_1 + R_2)\, 0 = 0$$

Appendix 4

When (public) firm 2 invests I in the new project, if we distinguish preinvestment variables by *, we obtain the new set of variables:

$$
(50)\quad
\begin{aligned}
\tilde{X}_2 &= \tilde{X}_2^* + I\tilde{Z} \\
\mu_2 &= \mu_2^* + I\mu_z \\
\sigma_{22} &= \sigma_{22}^* + 2I\,\sigma_{2z} + I^2\,\sigma_{zz} \\
\sigma_{12} &= \sigma_{12}^* + I\,\sigma_{1z} \\
\alpha_i &= \alpha_i \\
Z_i &= (1/\,\sigma_{11}^*)\left[\,(\mu_1^* - r\,p_1)\,\gamma_i - \sigma_{12}\alpha_i\right] \\
p_1 &= (1/r)\left[\mu_1^* - R\,(\sigma_{11}^* + \sigma_{12})\right]
\end{aligned}
$$

Notice that

$$
(51)\quad
\begin{aligned}
(d\,\mu_2\,/\,dI)\big|_{I=0} &= \mu_z \\
(d\,\sigma_{22}\,/\,dI)\big|_{I=0} &= 2\,\sigma_{2z} \\
(d\,\sigma_{12}\,/\,dI)\big|_{I=0} &= \sigma_{1z}
\end{aligned}
$$

As before

$$
(35)\quad (dU_i\,/\,dI)\big|_{I=0} = (dE_i\,/\,dI)\big|_{I=0} - (1\,/2\,\gamma_i)\,(dV_i\,/\,dI)\big|_{I=0} \qquad i=1,2
$$

but now

$$
\begin{aligned}
(52)\quad (dE_i\,/\,dI)\big|_{I=0} &= (d\left[\,r\left\{m_i^* + (Z_i^* - Z_i)\,p_1\right\} + \right. \\
&\quad \left. + Z_i\,\mu_1^* + \alpha_i\,\mu_2\right]/\,dI)\big|_{I=0} \\
&= (\mu_1^* - r\,p_1^*)\,(dZ_i\,/\,dI)\big|_{I=0} + \alpha_i\,\mu_z \qquad i=1,2 \\
(53)\quad (dV_i\,/\,dI)\big|_{I=0} &= (d\left[\,Z_i^2\,\sigma_{11}^* + \alpha_i^2\,\sigma_{22} + 2Z_i\,\alpha_i\,\sigma_{12}\right]/dI)\big|_{I=0} \\
&= 2\left[\alpha_i^2\,\sigma_{2z} + \alpha_i Z_i^*\,\sigma_{1z} + (\,\sigma_{11}^* Z_i^* + \alpha_i\,\sigma_{12}^*)\cdot\right. \\
&\quad \left.\cdot(\,d\,Z_i\,/\,dI)\big|_{I=0}\,\right] \qquad i=1,2
\end{aligned}
$$

Substituting (52) and (53) into (35) one gets

$$(46)\quad (dU_i / dI)\big|_{I=0} = \alpha_i \mu_z - (1/\beta_i)(\alpha_i^2 \sigma_{2z} + \alpha_i z_i^* \sigma_{1z}) + \\ + \left[\mu_1^* - r\,\rho_1^* - (1/\beta_i)(\sigma_{11}^* z_i^* + \alpha_i \sigma_{1z}^*)\right] \cdot \\ \cdot (dz_i / dI)\big|_{I=0} \\ = \alpha_i \mu_z - (1/\beta_i)(\alpha_i^2 \sigma_{2z} + z_i^* \alpha_i \sigma_{1z}) \quad i=1,2$$

since applying (21) the expression in brackets cancels out. (Q.E.D)

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